THE ELEMENTARY

GEOMETRY OF CONICS

WITH

A CHAPTER ON THE LINE INFINITY

BY

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SEVENTH EDITION REVISED
WITH A NEW TREATMENT OF THE HYPERBOLA

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PREFACE.

THE work as now revised includes (beside minor improvements and additions) *A New Treatment of the Hyperbola*, which was read to the Association for the Improvement of Geometrical Teaching in January 1890, and notes of *A Course for Beginners*

These are placed in separate Chapters XI. and XII respectively, so as not to disturb the order of the text as it stood in the sixth edition.

The figures for the new Articles *47, 84, 85 are by Mr G. T Bennett, Scholar of St John's College

C. TAYLOR

May 1891

PREFACE TO THE THIRD EDITION.

THE Geometry of Conics has been recast in the third edition, so as to serve as an introduction to a larger work now approaching completion

The characteristic feature of the edition is the use of the Eccentric Circle, which contributes to a concise and uniform treatment of the three species of conics

The Asymptotes of the hyperbola are shewn to be coincident with its self-conjugate diameters, and their properties are deduced from a limiting case of a property of conjugate diameters in general

The principle that Chord-properties should be proved independently of Tangent-properties is still adhered to, although in the general rearrangement of the text it seemed no longer desirable to confine the two classes of properties to separate chapters

The work now to some extent resembles my first work on *Geometrical Conics*, published in 1863, but the general chapter has been made more complete than I was then able to make it

October 1879

PREFACE TO THE FOURTH EDITION.

A FOURTH edition having been called for the work has been revised and in a measure enlarged by the insertion of some further corollaries (Arts 40, 54 59) and a chapter

on Curvature and also of a Lemma on points at infinity and a Scholium on the metric properties of diameters which are needful for a right conception of the hyperbola in itself and in its relation to the ellipse

The collection of Problems has been reconstructed by Mr J. S Yeo, Fellow of St John's College

A sketch of the history of this branch of Mathematics from the earliest times will be found in the Prolegomena to the work referred to in the preface to the third edition as approaching completion, and since published under the name of *An Introduction to the Ancient and Modern Geometry of Conics*

November 1883

PREFACE TO THE FIFTH EDITION.

THE fifth edition of the Elementary Geometry of Conics contains a chapter on the Line Infinity, the text of which gives a quasi-geometrical determination of the imaginary points at infinity through which all circles in a plane pass, while attention is called in a scholium to an apparent failure of the Cartesian method to shew that there are two such points only in any plane The subject is one which demands careful thought with some faculty of imagination in the student, and it is not intended that the chapter should as a rule be read for the first time without the help of a teacher

July 1888.

PREFACE TO THE SIXTH EDITION

In the sixth edition a construction for tangents to a conic by the Eccentric Circle has been added to Art 6, and Art 16 is applied to the special case of chords of a hyperbola regarded as cut by an asymptote (Art 51) The simplicity of this treatment of asymptotes is a further advantage accruing from the use of the Eccentric Circle

To one who is not familiar with the different forms of the conics the mere generality of a proof which applies to all of them may be found perplexing In such case the student may be advised to begin with the chapter on the parabola, supplemented by one or other of the constructions for tangents, and in his first reading of Chapters II and IV to regard them as chapters on the ellipse only, passing over all references to the hyperbola Or he may begin with the chapter on the cone

August 1889

CONTENTS.

THE GEOMETRY OF CONICS.

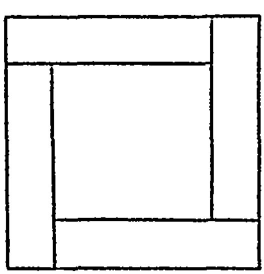
LEMMAS

We shall have occasion in the course of the work to assume the following Lemmas. It will suffice to refer to them one by one as they are wanted.

A. *To prove geometrically that*

$$(a+b)^2-(a\sim b)^2=4ab$$

If four rectangles whose sides are equal to a and b be fitted symmetrically about the square on $a\sim b$, the whole figure will make up the square on $a+b$.



Therefore $\qquad (a+b)^2=(a\sim b)^2+4ab,$

or $\qquad (a+b)^2-(a\sim b)^2=4ab.$

The same is proved in Euclid II 8, but by an unsymmetrical construction which shews only a gnomon of equal area instead of the four rectangles

In illustration of the above, let PQ be any straight line bisected in O, and let Y be any point in PQ produced (Figure Art 13), then

$$PY^2 - QY^2 = (OY + OP)^2 - (OY - OP)^2$$
$$= 4OY.OP = 2OY\ PQ^*$$

B *The distance of any point in a straight line produced or within it from the middle point of the line is half the sum or difference of its distances from the extremities of the line*

For let mM be any straight line and L its middle point And first let a point R be taken in mM produced. Then

$$RL - RM = \tfrac{1}{2}mM = Rm - RL,$$

or

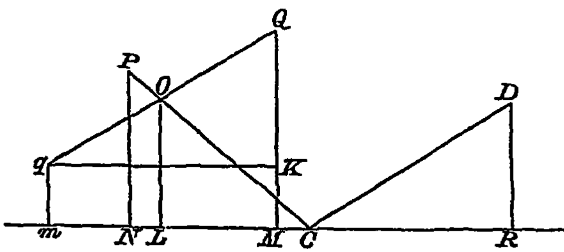
$$RL = \tfrac{1}{2}(RM + Rm).$$

Next let a point N be taken within mM Then

$$NM - NL = \tfrac{1}{2}mM = Nm + NL,$$

or

$$NL = \tfrac{1}{2}(NM \sim Nm)$$



C *The ordinate of the middle point of a straight line is equal to half the sum or difference of the ordinates of its extremities*

From the extremities of a straight line Qq and from its middle point O let parallels QM, qm, OL be drawn to meet

* Otherwise thus Since $PY + QY = 2OY$ (Lemma B), and $PY - QY = PQ$; therefore (Euc II 5, Cor) $PY^2 - QY^2 = 2OY\ PQ$

any given straight line or *axis;* and let these parallels be called the *ordinates* of the points from which they are drawn

Draw qK parallel to mM to meet QM, and first let Q and q lie on the same side of the axis Then, since qK cuts off from OL a length equal to $\frac{1}{2}QK$, therefore

$$OL - qm = \tfrac{1}{2}(QM - qm),$$

or
$$OL = \tfrac{1}{2}(QM + qm)$$

Next draw a figure in which Q and q lie on opposite sides of the axis Then it may be shewn in like manner that

$$OL = \tfrac{1}{2}(QM \sim qm)$$

That is to say, the ordinate of O is equal to half the sum or difference of the ordinates of Q and q according as these points lie on the same side or on opposite sides of the axis

D *The sum of the squares of the distances of any point from the extremities of any straight line is double of the sum of the squares of its distance from the middle point of the line and of half the line*

For if SS' be the given straight line, P the given point, PN a perpendicular to SS', and C the middle point of SS',

then
$$SP^2 = CS^2 + CP^2 + 2CS \cdot CN,$$

and
$$S'P^2 = CS'^2 + CP^2 - 2CS' \cdot CN,$$

therefore by addition, since CS' is equal to CS,

$$SP^2 + S'P^2 = 2CS^2 + 2CP^2$$

E *To divide a given straight line in a given ratio of majority or of minority*

A ratio is said to be a ratio of equality, majority or minority according as it is equal to unity, or greater or less than unity

(1) First let SX be the given straight line, which is to be divided in a given ratio of majority Draw SH in any direction, and produce it to K, so that $SH : HK$ may be equal to the given ratio Join KX, and draw HA parallel

to KX to meet SX in A Then since

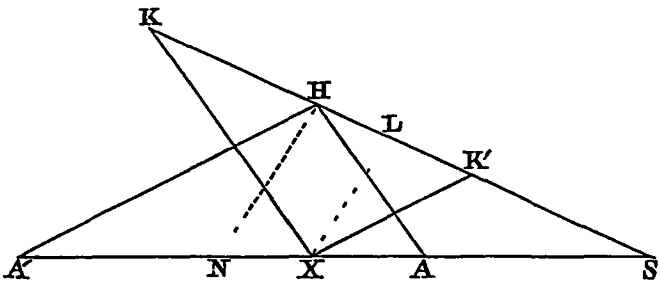
$$SA \quad AX = SH \quad HK, \qquad \text{[Euc VI. 2}$$

the point A divides SX internally in the given ratio

Upon HS take HK' equal to HK Join $K'X$, and draw HA' parallel to it to meet SX produced in A' Then

$$SA' \quad A'X = SH \quad HK' = SH \quad HK,$$

or the point A' divides SX externally in the given ratio



In this case the points A and A' will always lie on the *same side* of S, because K and K' lie on the same side of S

(ii) Next let it be required to divide SX in a given ratio of minority SH HK Draw a figure in which HK is greater than SH Then, using the same lines of construction as before, we determine the two required points of division A and A', which must always lie on *opposite sides* of S, because K and K' lie on opposite sides of S

It is evident that there is only one point at a finite distance which bisects a straight line, or divides it in a ratio of *equality* But the point ∞ at infinity on any straight line SX likewise divides it in a ratio of equality $S\infty$ $X\infty$

F *To divide a given straight line in a ratio greater or less than a given ratio*

In the figure given above let it be required to divide SX in a ratio greater than the ratio of *majority* SA AX This is done by taking the point of division N anywhere between

A and A'. For if XL be drawn as in the figure parallel to NH to meet SH in L, then

$$\begin{aligned} SN : NX = SH \cdot HL &> SH : HK \\ &> SA \cdot AX \end{aligned}$$

In like manner it may be shewn that every point in AA' produced either way divides SX in a ratio less than $SA : AX$

Next, if $SA : AX$ be a ratio of *minority*, as in the second case of Lemma E, it may be shewn in like manner that SX is divided in a ratio less than $SA : AX$ by every point in AA', and in a ratio greater than $SA : AX$ by every point in AA' produced either way.

G. *Harmonic section of a straight line*

If a straight line SX be divided internally and externally in the same ratio at A and A' so that

$$SA' : A'X = SA : AX;$$

then
$$\begin{aligned} SA' : SA &= A'X : AX \\ &= SA' \sim SX . SX \sim SA, \end{aligned}$$

or SA', SX, SA are in harmonic progression.

Hence SX is said to be divided harmonically at A and A'

The relation between SA', SX, SA may also be written in the form

$$\frac{1}{SA} + \frac{1}{SA'} = \frac{2}{SX}.$$

Notice, as a limiting case, that SX is divided harmonically by its middle point and its point at infinity, for if A' be taken at infinity SA becomes equal to $\frac{1}{2}SX$.

H. *Opposite points at infinity coincide*

This Lemma is necessary for the right understanding of the genesis of the hyperbola

Take an unlimited straight line PP', and let OM be the perpendicular to it from an assumed point O without it

The line may be regarded as traced by a point P lying on a ray OP which turns continuously about O For adjacent positions P and P' of the tracing point the angle of rotation POP' is small conversely we may say that the smallness of this angle is the test of the adjacence of P and P'

But if P and P' be on opposite sides of and indefinitely remote from M, we may still pass, viz through infinity, from P to P' by turning OP through an indefinitely small angle Such points are therefore quasi-adjacent, and the opposite points at infinity on the line are quasi-coincident

From this it follows that every straight line, or system of parallels, has one point only at infinity, as was assumed in Lemmas E and G

DEFINITIONS.

[Further definitions will be given at the beginnings of the chapters, as occasion arises.]

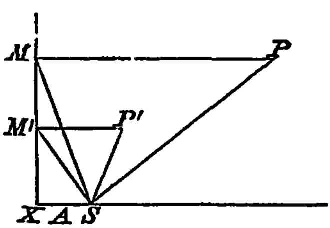
1. A Conic Section*, or briefly a *Conic*, is the curve traced in a plane by a point which moves in such a way that its distance from a given point is in a constant ratio to its perpendicular distance from a given straight line. The given point is called the *Focus*, the given straight line the *Directrix*, and the constant ratio the *Eccentricity* of the conic.

If S be the focus, P any point on the conic and PM the perpendicular from it to the directrix, the ratio of SP to PM is constant. If P' be any other point on the conic, and $P'M'$ the perpendicular from it to the directrix, then

$$SP : PM = SP' : P'M',$$

or

$$SP : SP' = PM : P'M'.$$



Let us now take a particular case, and suppose the directrix to be at an infinite distance from the focus. In this case $PM : P'M'$ is a ratio of equality, and therefore $SP . SP'$ is

* The conic sections were so called because they are the curves in which a plane can be made to intersect a cone, as will be shewn in the seventh chapter.

a ratio of equality That is to say, SP is always equal to SP', and the locus of P is a circle Thus it appears that our definition of a conic is an extension of the definition of a circle

2 A conic is called a *Parabola*, an *Ellipse*, or a *Hyperbola*, according as its eccentricity is a ratio of equality, of minority, or of majority

3 The *Axis* is the unlimited straight line through the focus at right angles to the directrix, and the points in which it meets the conic are called the *Vertices* When one vertex only is spoken of the vertex which lies between the focus and the directrix is signified

It is evident from Lemma E that the parabola has only one vertex at a finite distance, and that the ellipse and the hyperbola have each two vertices.

4 The middle point of the line joining the vertices is called the *Centre* of the conic The ellipse and the hyperbola are called *Central Conics*, in contrast with the parabola which has no centre at a finite distance The straight line through the centre at right angles to the axis is called the *Conjugate Axis*

5 A *Chord* of a conic is properly the finite straight line joining any two points on the curve but the term is also used to denote the unlimited straight line joining any two points on the curve The extremities of a chord are the points in which it meets the conic

6 The *Latus Rectum* is the focal chord, or chord through the focus, at right angles to the axis

7 A *Diameter* is the locus of the middle points of a system of parallel chords it will be proved that the diameters of conics are straight lines One diameter is said to be *conjugate* to another when it bisects chords parallel thereto

8 The *Principal Ordinate*, or briefly the *Ordinate*, of any point is the perpendicular drawn from it to the axis More generally, the ordinate of any point *to any diameter* is

the line drawn from the point to that diameter in the direction parallel to the conjugate diameter.

9. A *Tangent* to a conic is the limiting position of a chord or secant whose two points of intersection with the curve have become coincident Thus if P and Q be adjacent points on a conic, and if the chord joining them be turned round P, or be moved about in any other way, until its extremity Q coincides with P, the chord in its limiting position becomes the tangent at P Hence a tangent is said to be a straight line which passes through two *consecutive* or *coincident* points on the curve

The chord of contact of two tangents is the chord joining their points of contact

10 The *Normal* at any point of a conic is the perpendicular to the tangent at that point

11. If about any point in the plane of a conic a circle be described such that the ratio of its radius to the perpendicular distance of its centre from the directrix is equal to the eccentricity, the circle may be called the eccentric circle of the conic with respect to that point, or briefly the *Eccentric Circle of the Point*

12 The *Order* or *Degree* of a curve is determined by the number of points in which it can be met by a straight line Thus a curve of the second order or degree is one which a straight line meets generally in two and never in more than two points

All the points at infinity in any plane constitute a locus of the first degree, which is called the *Straight line at Infinity* or the *Line Infinity*, since by Lemma H every other straight line in the plane passes through one point only at infinity.

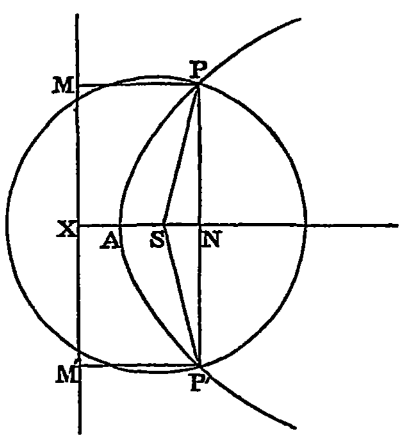
CHAPTER I.

DESCRIPTION OF THE CURVE.

1 *Having given the focus, directrix and eccentricity of a conic, it is required to describe the curve.*

Let S be the focus*, MM' the directrix, and X the point in which the axis meets the directrix In SX take the point A so that the ratio of SA to AX may be equal to the eccentricity (Lemma E) Then A is the vertex of the conic.

Draw a straight line cutting the axis at right angles in N, and let P and P' be the points in which the line meets



* The planets describe approximately ellipses about the sun in one focus For this reason this first letter of *Sol* is used, as by Newton, to denote the *Focus*, or as he called it the Umbilicus We shall use the letters S, A, X as above without further explanation, so that $SA : AX$ will always denote the eccentricity

the circle described with S as centre and radius SP, such that

$$SP : NX = SA \,.\, AX.$$

Draw PM perpendicular to the directrix. Then

$$SP \quad PM = SP \cdot NX = SA \cdot AX,$$

or P is a point on the conic In like manner it may be shewn that P' is a point on the conic

If now we suppose the chord PP' to slide at right angles to the axis, so as to assume all *possible* positions, its extremities will trace out the complete curve.

2 *The three species of conics*

In order that the line and the circle in the above construction may intersect, the length SN must be less than the radius SP, or

$$SN : NX < SA \cdot AX$$

In the case of the *Parabola* we must have $SN < NX$ The point N may therefore be taken anywhere in XA produced, and the curve consists of one infinite branch spreading out from the vertex and away from the directrix

In the *Ellipse*, if A' be the second vertex, the point N may be taken anywhere between A and A' (Lemma F), and the curve consists of one oval branch (Fig Art 5) lying on the same side of the directrix with the focus

In the *Hyperbola*, if A' be the second vertex, the point N may be taken anywhere in AA' produced either way (Lemma F), and the curve consists of two infinite branches on opposite sides of the directrix

3 *The symmetry of the curve.*

From the foregoing construction it is evident that the curve is symmetrical with respect to its axis, since its points are always determined in pairs as P and P' in corresponding positions above and below the axis, so that the part of the curve below the axis is the accurate reflexion of the part above the axis.

It follows also that the tangent at A is at right angles to the axis, since when the point N coincides with the vertex $SP = SA = SP'$, that is to say, the points P and P' coalesce at A, and the chord joining them, which is always at right angles to the axis, becomes the tangent to the conic at its vertex (Def. 9)

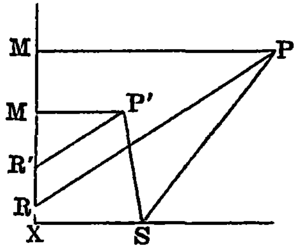
4 *The focal distance of any point on a conic is in a constant ratio to the distance of the point from the directrix measured parallel to any fixed straight line which meets the directrix*

From any two points P and P' on the conic draw PR and $P'R'$ in any fixed direction, to meet the directrix, and draw PM and $P'M'$ perpendicular to the directrix

Then $$SP\ \ PM = SP'\ \ P'M', \qquad \text{[Def 1}$$

and $$PM\ \ PR = P'M'\ \ P'R',$$

by similar triangles



Therefore $$SP\ \ PR = SP'\ \ P'R',$$

or (since we may consider P' and $P'R'$ to remain fixed whilst P varies) the focal distance SP varies as the distance PR to the directrix measured in any given direction

Conversely, every point P which satisfies the above relation is a point on the conic

Notice in particular that if any chord PQ meet the directrix in R,

$$SP\ \ PR = SQ\ \ QR.$$

5 *A conic is a curve of the second order.*

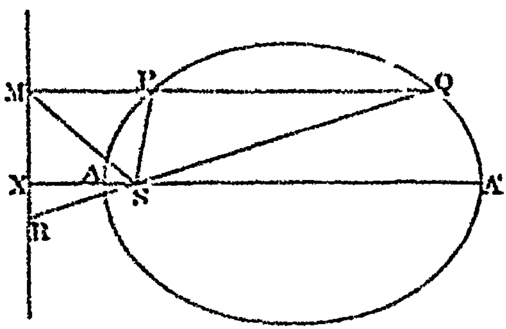
Let a straight line drawn in any direction cut the conic in P and meet the directrix in M. Make the angle MSR equal to MSP, and let the line RS produced meet MP in Q.

Then, by Euclid VI. A or 3, since SM bisects the angle PSQ or its supplement,

$$SQ : SP = QM : PM,$$

or

$$SQ : QM = SP : PM,$$



and therefore Q is a point on the curve (Art 4); and it is evident that no third point of intersection of the line PQ with the conic can be determined

It follows that a straight line which meets a conic will in general meet it in two points, and never in more than two A conic is therefore a curve of the second order or degree (Def 12)

In the *Ellipse* P and Q always lie on the same side of M.

In the *Parabola*, if PM be parallel to the axis and therefore equal to SP,

$$\angle MSR = MSP = SMP = MSX,$$

or SR coincides with SX and the point Q recedes to infinity. Hence every straight line parallel to the axis of a parabola meets the curve in one point only at a finite distance.

6 *To describe a conic of given focus, directrix and eccentricity, and to draw tangents to it, by means of the eccentric circle*

Describe the eccentric circle of any point O* in the plane of the conic (Def 11), and let a straight line through S meet the circle in p and the directrix in R

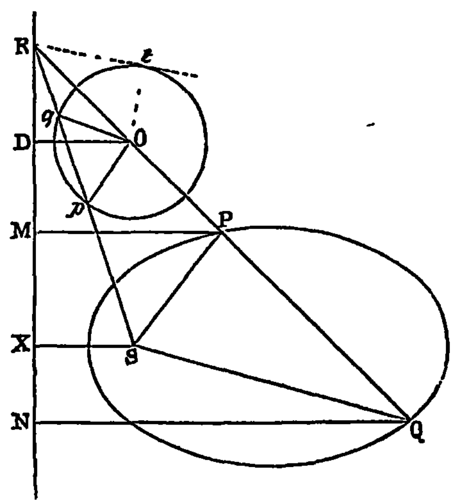
Let the focal radius parallel to pO meet RO in P, and let OD and PM be perpendiculars to the directrix

Then since SP, pO and PM, OD are parallels,

$$SP : Op = PR : OR$$
$$= PM : OD,$$

or

$$SP : PM = Op : OD$$
$$= \text{the eccentricity}$$



Hence, as p moves round the circle, P traces the conic which was to be described

Conversely, if any point O be taken on a chord PQ of a conic, the eccentric circle of O will meet SR (drawn to the

* Let SL be the semi-latus rectum, and let the ordinate of O meet the axis in N and XL in K Then since $KN : OD = KN : NX = SL : SX = SA : AX$ the radius of the circle must be taken equal to KN

point of concourse R of the chord with the directrix) in points p and q lying upon radii parallel to SP and SQ

The student should now draw figures in which the eccentric circle touches or cuts the directrix, and trace the corresponding conics, which will be in the one case parabolas and in the other hyperbolas

When the points p and q coalesce, the points P and Q coalesce Hence the following construction for the tangents to the conic from a given point O.

Let either tangent from S to the circle meet the directrix in R, then RO is a tangent to the conic

The tangent Sp to the circle and the tangent OP to the conic subtend equal angles at O and S respectively But the two tangents from S to the circle subtend equal angles at O Therefore the two from O to the conic subtend equal (or supplementary) angles at S

For the special case of the parabola the definition of the eccentric circle is not required, the construction for the tangents from a given point O being simply, *with O as centre describe a circle touching the directrix, and let the tangents to it from S meet the directrix in R and R', then RO and $R'O$ are the required tangents to the parabola*

CHAPTER II

THE GENERAL CONIC

[*For a course of reading to be preferred in some cases see Chap XII*]

We shall commence by proving some of the principal properties which are common to the parabola, the ellipse and the hyperbola.

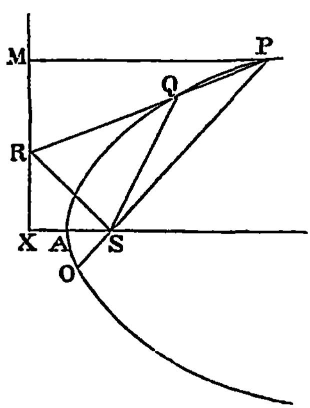
The Tangent

PROPOSITION I.

7. *Each of the two tangents which can be drawn to a conic from any point on its directrix subtends a right angle at the focus*

Let P and Q be adjacent points on the curve, and let PQ produced meet the directrix in R Then it may be shewn (Art. 4) that

$$SP : SQ = PR : QR,$$



and therefore SR bisects the angle which SQ makes with PS produced [Euc VI. A

Let PS meet the curve again in O Then since the angles RSQ and RSO are always equal, therefore in the limit, when Q coalesces with P, each of them becomes a right angle, and RP, which becomes the tangent at P (Def 9), subtends a right angle at S In like manner it may be shewn that the other tangent which can be drawn to the conic from R subtends a right angle at S

Corollary

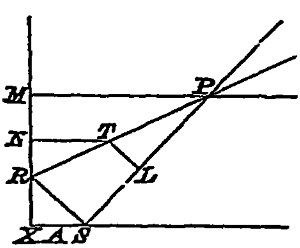
Hence it appears that the tangents at the extremities of any focal chord OP meet at a point R lying on the directrix, and such that SR is at right angles to OP Conversely, if tangents be drawn to a conic from any point R on the directrix their chord of contact will be the focal chord at right angles to RS The tangents at the extremities of the latus rectum meet at X

PROPOSITION II.

8 *If from any point* T *on the tangent at* P *there be drawn perpendiculars* TL *and* TN *to* SP *and the directrix, the ratio of* SL *to* TN *will be constant and equal to the eccentricity*

For if the tangent at P meet the directrix in R, and if PM be a perpendicular to the directrix, then since SR is at right angles to SP (Prop I.) and is therefore parallel to TL, it follows that

$$SL : SP = TR : PR$$
$$= TN : PM$$



Therefore $$SL \quad TN = SP . PM$$
$$= SA : AX,$$

or SL is equal to the radius of the eccentric circle of T.

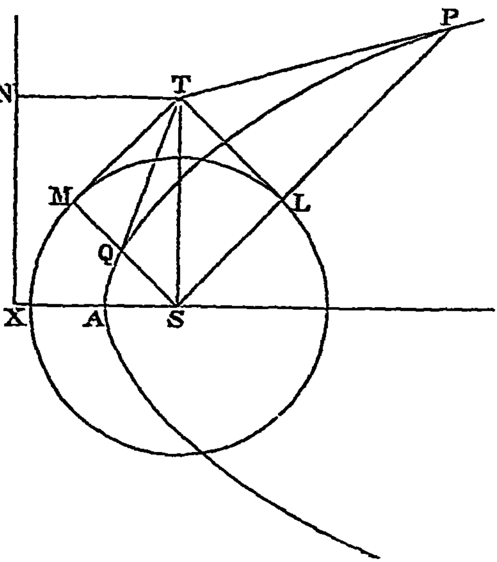
Corollary

To draw a pair of tangents to a conic from a given external point T, with S as centre describe a circle equal to the eccentric circle of T, and draw the tangents TL and TM to the circle (Fig Art 9); then by the converse of the proposition SL and SM will pass through the points of contact P and Q of the required tangents Draw SR at right angles to SL to meet the directrix in R, then TR is one of the two tangents Draw SR' at right angles to SM to meet the directrix in R', then TR' is the second tangent from T For a construction by the eccentric circle see Art 6

PROPOSITION III.

9 *The two tangents which can be drawn to a conic from any external point subtend equal or supplementary angles at the focus*

For if TP and TQ be the two tangents to a conic from



the point T, and TL, TM, TN be perpendiculars to SP, SQ and the directrix, then by Prop II, since T lies on the tangent at P,

$$SL \cdot TN = SA \cdot AX,$$

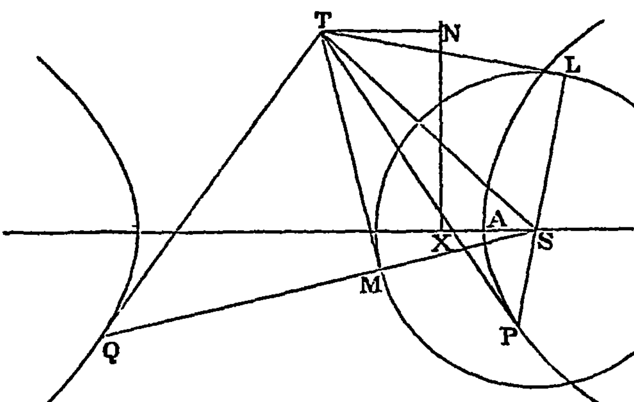
and since T lies on the tangent at Q,

$$SM \cdot TN = SA \cdot AX$$

Therefore in the right-angled triangles STL and STM the side SL is equal to SM, and the hypotenuse ST is common, and therefore the angle TSL is equal to TSM

Now (i) if TP and TQ touch the same branch of the conic, the angles which they subtend at S are either *equal* to TSL and TSM (as in the above figure) or *supplementary* thereto In either case the two tangents subtend EQUAL angles at S

But (ii) if TP and TQ touch opposite branches of a hyperbola (as in the next figure), so that one and one only of



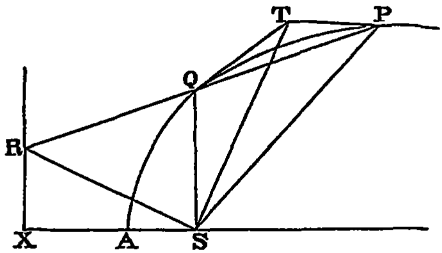
the lines SL and SM has to be produced backwards to P or Q, the two tangents will subtend SUPPLEMENTARY angles at S.

PROPOSITION IV

10 *The point of concourse of any two tangents to a conic and the point in which their chord of contact meets the*

directrix lie upon a pair of focal radii which include a right angle.

If P and Q be points on the same branch of a conic, and if PQ meet the directrix in R, then, as in Prop I, SR bisects



the angle which SQ makes with PS produced, that is to say, it bisects the supplement of PSQ.

Also, if the tangents at P and Q meet in T, the line ST bisects the angle PSQ (Prop III)

Therefore ST and SR bisect supplementary angles, and are therefore at right angles to one another

If TP and TQ touch opposite branches of a hyperbola, then (completing the second figure of Art 9) it may be shewn that in this case also the angle TSR is a right angle

The Normal.

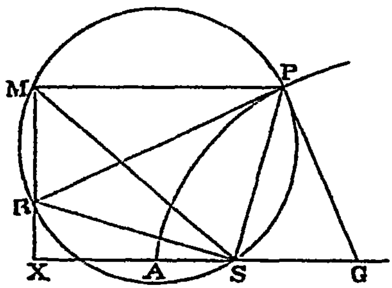
PROPOSITION V

11 *The normal at any point of a conic meets the axis at a distance from the focus which is to the focal distance of the point in a constant ratio equal to the eccentricity*

For if the tangent at P meet the directrix in R, the circle on PR as diameter will pass through S, since PSR is a right angle, and it will likewise pass through M, the projection of P upon the directrix, and the normal at P will touch the circle, since it is at right angles to its diameter PR.

Let the normal meet the axis in G

Then $$\angle SPG = SMP,$$
in the alternate segment of the circle, and
$$\angle PSG = SPM,$$
by parallels



Therefore the triangles SPG and SPM are similar, and
$$SG : SP = SP \quad PM$$
$$= SA \cdot AX,$$
or SG varies as SP, as was to be proved.

Conversely, if in AS produced there be taken a point G such that
$$SG \quad SP = SA \,.\, AX,$$
the line PG will be the normal at P

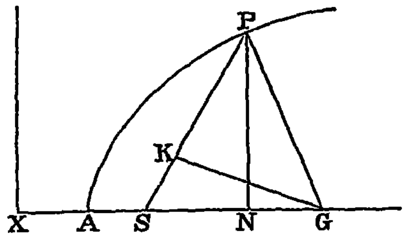
PROPOSITION VI

12 *At any point of a conic the projection of the normal (terminated by the axis) upon the focal radius is equal to the semi-latus rectum*

Let the normal at P meet the axis in G draw GK perpendicular to $SP\cdot$ and draw PN perpendicular to the axis

Then by similar right-angled triangles SKG and SNP,
$$SK \,.\, SN = SG \cdot SP$$
$$= SP \quad PM \qquad \text{[Prop v}$$
$$= SP \quad NX$$

Hence $SP-SK \quad NX-SN=SP \quad NX$,



that is to say, PK is to SX in a constant ratio equal to the eccentricity, and is therefore equal to the semi-latus rectum

Note that when P is taken at an extremity of the latus rectum PK coalesces with the semi-latus rectum

Diameters.

PROPOSITION VII

13 *The locus of the middle points of any system of parallel chords of a conic is a straight line which meets the directrix on the straight line through the focus at right angles to the chords*

Let PQ be any one of a system of parallel chords, V the point in which the focal perpendicular upon them meets the directrix, R and Y the points in which PQ meets the directrix and SV respectively

Then since $SP \cdot PR=SQ \quad QR$, [Art 4

therefore (supposing for example that SP is greater than SQ)

$$SP^2-SQ^2 \quad PR^2-QR^2=SP^2 \quad PR^2,$$

or, subtracting SY^2 from SP^2 and SQ^2 respectively,

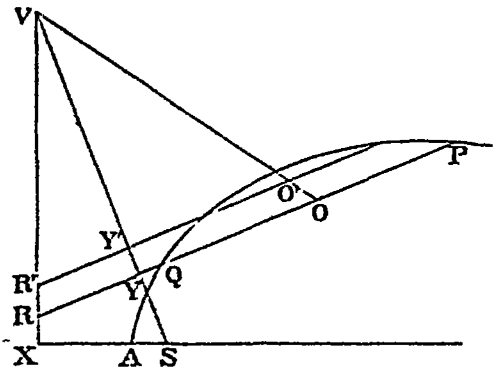
$$PY^2-QY^2 \quad PR^2-QR^2=SP^2 \quad PR^2$$

[Euc I 47

But if O be the middle point of PQ,

$$PY^2 - QY^2 = 2OY \cdot PQ, \qquad \text{[Lemma A}$$

and similarly, $$PR^2 - QR^2 = 2OR.PQ$$



Hence $$OY \cdot OR = SP^2 \cdot PR^2,$$

which (Art 4) is a *constant ratio* so long as PQ is drawn in a specified direction.

Hence if any other chord be drawn parallel to PQ, and if O', Y', R' be the new positions of O, Y, R, it follows that

$$O'Y'.O'R' = OY \cdot OR,$$

and hence that the points O, O', V lie in a straight line*

If now we suppose the point O' to remain fixed whilst PQ moves parallel to itself, the point O will always lie upon a *fixed* straight line $O'V$, as was to be proved

Proof by the eccentric circle

The eccentric circles of P, Q meet in S and have SY for radical axis A common tangent MM' to the circles passes through R, makes a constant angle with PQ (one of a series of parallel chords), and is bisected by SY in a point at which OR subtends a right angle Hence $OY \cdot YR$ is a constant ratio, OVS a constant angle, and OV a fixed line In the parabola (Art 20) the directrix is a common tangent to the circles

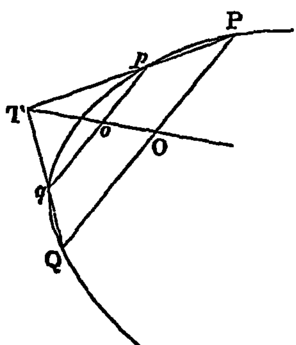
* For if they do lie on a straight line, $O'Y' \cdot OY = O'V \cdot OV = O'R' \cdot OR$ The required converse may be easily deduced by a *reductio ad absurdum*

Corollary 1

The tangent at either extremity of a diameter is parallel to the chords which the diameter bisects, since any one of the bisected chords may be supposed to move parallel to itself until its segments vanish together and its extremities coalesce, so that it becomes a tangent, viz at an extremity of the bisecting diameter If a diameter meets the conic in two points, the tangents at those points are parallel to the *ordinates* (Def. 8) of that diameter and to one another

Corollary 2.

If PQ and pq be any two of a system of parallel chords, and O and o be their middle points, which will lie on a fixed diameter, it is evident that Pp and Qq will meet at a point T lying on that diameter Hence, making pq move



parallel to itself until it coalesces with PQ, so that TP and TQ become the tangents at P and Q, we see that *the tangents at the extremities of any chord meet upon the diameter which bisects the chord*, and conversely, that the diameter to any external point bisects the chord of contact of the two tangents from that point

PROPOSITION VIII.

14 *Every central conic is divided symmetrically by its conjugate axis, and has a second focus and directrix*

Let AA' be the transverse axis of a central conic, PQ any chord parallel thereto, and M the point in which PQ or its prolongation meets the directrix

Bisect PQ in O, and draw SY perpendicular to PQ, then it may be shewn, precisely as in Prop. VII., that

$$OY : OM = SP^2 : PM^2, \qquad \text{[Fig. Art. 37}$$

and hence that the locus of O is a straight line at right angles to AA'.

Since this straight line must also bisect AA' (which is a limiting position of the chord PQ), it meets AA' at the centre C of the conic, and coincides with the *conjugate axis* (Def. 4)

It is evident that this line divides the curve into two parts such that each is the exact reflexion of the other, and hence that the curve has a second focus H and directrix NW, the exact counterparts of the original focus and directrix with reference to which the conic was defined

Corollary 1

From the symmetry of a central conic with respect to its two axes, it is manifest that every chord through its centre is bisected at that point, and hence that all diameters pass through the centre It is further evident that any two diameters or focal chords equally inclined to either axis are equal to one another, and that any two tangents to the conic from a point on either axis are likewise equal.

Corollary 2.

In Art 13 let the diameter parallel to PQ meet the directrix in V'. Then in the triangle CVV', since VS is at right angles to CV' and CS to VV', therefore S is the orthocentre and $V'S$ is at right angles to CV* Hence the diameter CV' bisects chords parallel to CV, or if one diameter be conjugate to a second, the second is conjugate to the first

* The three perpendiculars of a triangle meet in a point, which is called its *orthocentre*.

The Segments of Chords.

PROPOSITION IX

15 *The semi-latus rectum is a harmonic mean between the segments of any focal chord*

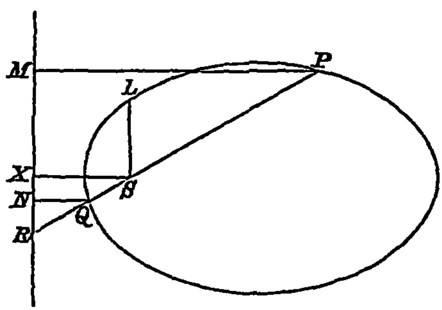
Let a focal chord PQ or its prolongation meet the directrix in R, and let PM, SX, QN be perpendiculars to the directrix

Then, since

$$SP \quad SQ = PM \quad QN = PR \cdot QR,$$

therefore $$PR - SR \quad SR - QR = PR \quad QR,$$

or PR, SR, QR are in harmonic progression. [Lemma G



But by parallels, and from the definition of the curve, if SL be the semi-latus rectum,

$$PR \quad SR \quad QR = PM \quad SX \cdot QN$$
$$= SP \quad SL \quad SQ$$

Therefore also SP, SL, SQ are in harmonic progression

Corollary

It is easy to deduce that

$$SL \quad PQ = SL\,(SP + SQ) = 2SP \quad SQ$$

Hence, if pq be any other focal chord,

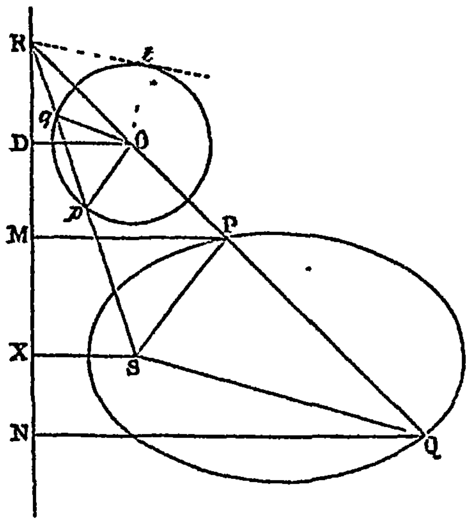
$$PQ : pq = SP \cdot SQ : Sp \cdot SQ,$$

or *focal chords are to one another as the rectangles contained by their segments*

PROPOSITION X.

16 *A chord of a conic being divided at any point, to determine the rectangle contained by its segments*

Let a chord PQ of a conic be divided within or without at any point O, and let it meet the directrix in R Then, if SK and Rt be tangents to the eccentric circle of O, the rectangle $OP.OQ$ will be to SK^2 as OR^2 to Rt^2.



Let parallels through O to SP and SQ meet OR in p and q, and let OD and PM be perpendiculars to the directrix

Then $\quad Op \cdot SP = OR . PR = OD \cdot PM,$

or $\quad Op \cdot OD = SP \cdot PM =$ the eccentricity,

and therefore p is on the eccentric circle of O

Similarly q is on the same circle.

Also $$OP\ Sp = OR\ Rp,$$
and $$OQ\ Sp = OR\ Rq$$
Hence $$OP\ OQ\ \ Sp\ Sq = OR^2\ \ Rp\ Rq,$$
or $$OP\ OQ\ \ SK^2 = RO^2\ \ Rt^2.\text{*}$$ [Euc III 36.

Corollary 1

The ratio $OP\ OQ\ \ SK^2$ depends only on the direction of PQ For when the angle ROD is given, RO is in a constant ratio to OD, and therefore to Ot Therefore also the angle ROt is given and $RO\ \ Rt$ is constant

Similarly, if $P'Q'$ be any second chord through O, the ratio $OP'\ OQ'\ \ SK^2$ depends only on the direction of $P'Q'$

Hence, compounding, $OP\ OQ\ \ OP'\ OQ'$ depends only upon the directions of PQ and $P'Q'$, and not upon the position of O That is to say.

The ratio of the rectangles contained by the segments of any two intersecting chords of a conic is the same as for any other two chords parallel to the former, each to each

Corollary 2

This ratio is equal to that of the parallel focal chords (Art 15, Cor), and in a central conic to that of the squares of the semi-diameters parallel to the chords, and in the general conic to the ratio of the squares of any pair of tangents parallel to the chords, since a tangent is defined as a chord whose extremities are coincident

* No real tangent can be drawn from S to the circle when O lies within the conic, nor from R when PQ meets both branches of a hyperbola. But $Sp\ Sq$ or $Rp\ Rq$ is then equal to the square of half the chord bisected at S or R (Euc III 35), and in all cases, if r be the radius of the circle,

$$OP\ OQ\ \ SO^2 \sim r^2 = RO^2\ \ RO^2 \sim r^2$$

If e be the eccentricity, $RO^2 \sim r^2\ \ RO^2 = 1 \sim e^2 . \cos^2 ROD$

Corollary 3

If a circle and a conic intersect in four points, their common chords are equally inclined, in opposite pairs, to the axis of the conic. For if POQ and pOq be one of the three pairs of common chords of a circle and a conic, the rectangles $PO.OQ$ and $pO\ Oq$ are as the focal chords parallel to PQ and pq (Cor 2)*, and the same rectangles are equal to one another by a property of the circle Therefore the focal chords parallel to PQ and pq are equal, and are therefore equally inclined to the axis [Art 15, Cor

Corollary 4.

If $Sp\ Sq$ were constant for a series of positions of O, the rectangle $OP\ OQ$ would vary only with the direction of PQ This condition is satisfied when the conic is a hyperbola and O any point on either asymptote (Art 48, end) It follows that, when O is any such point, $OP\ OQ$ is equal to the square of the parallel tangent (terminated by the asymptote), or of the parallel semi-diameter.

* In the central conics it is simpler to say, that the rectangles are *as the squares of the diameters* parallel to PQ and pq, and that equal diameters are equally inclined to the axis (Art 14, Cor 1) Or we may use the parallel tangents in all cases

CHAPTER III

THE PARABOLA

17 THE annexed further definitions will be required in the present chapter

The principal *Abscissa* or *Absciss* of any point with respect to a parabola is the finite segment cut off from the axis by the principal ordinate of the point The abscissa or absciss of a point *to any diameter* is the finite segment cut off from it by the ordinate of the point to that diameter [Def 8

The *Parameter* of any diameter of a parabola is the focal chord which it bisects thus the latus rectum is the parameter of the axis

The *Subtangent* at any point of a conic is the intercept upon the axis between the tangent and the ordinate of the point, and the *Subnormal* is the intercept between the normal and the ordinate The subtangent *to any diameter* is the intercept thereupon between the tangent and the ordinate of its point of contact to that diameter

18 The eccentricity of the parabola being a ratio of equality, the semi-latus rectum is equal to SX, and therefore to $2SA$ Also in the parabola SG becomes equal to SP (Arts 11, 25), and the eccentric circles of all points (Def 11) touch the directrix

The property of diameters in Art 13 has been proved for all conics without distinction, but we shall also shew that it can be proved with peculiar ease for the special case of the parabola.

Chord-Properties*.

PROPOSITION I.

19. *The principal ordinate of any point on a parabola is a mean proportional to its abscissa and the latus rectum*

If PN and AN be the principal ordinate and abscissa of any point P on the parabola, we have to shew that

$$PN^2 = 4AS \cdot AN$$

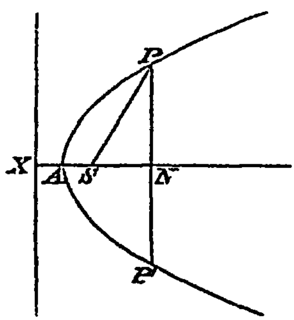
By Euclid I. 47 and from the definition of the parabola,

$$PN^2 + SN^2 = SP^2 = NX^2.$$

Hence (taking for example the case in which AN is greater than AS),

$$\begin{aligned} PN^2 + (AN - AS)^2 &= (AN + AS)^2 \\ &= (AN - AS)^2 + 4AS \cdot AN. \end{aligned}$$

[Lemma A



Therefore PN^2 is equal to $4AS \cdot AN$, or in other words, PN is a mean proportional to the abscissa AN and the latus rectum (Art. 18).

* By chord-properties (as distinguished from tangent-properties) we understand such properties as do not presuppose the definition of a tangent (Def 9) It is desirable to avoid using tangent-properties to prove chord-properties, the reverse order being the only natural one, so long as we regard a tangent as the limit of a chord

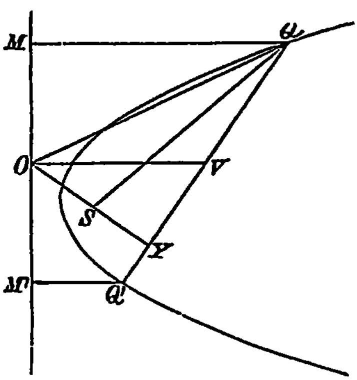
Conversely, if PN and AN be thus related, the locus of P will be a parabola of latus rectum equal to $4AS$.

PROPOSITION II. V

20 *The locus of the middle points of any number of parallel chords of a parabola is a straight line parallel to the axis, and the bisecting line meets the directrix on the straight line through the focus at right angles to the common direction of the chords*

Let QQ' be any one of a system of parallel chords, and let QM and $Q'M'$ be perpendiculars to the directrix.

Let the focal perpendicular upon the chords meet QQ' in Y and the directrix in O, and let the parallel to the axis through O (which is a *fixed* straight line) meet QQ' in V Then will V be the middle point of QQ'



For by Euclid I. 47 and from the definition of the parabola (taking the case in which Y lies in OS produced),

$$OM^2 = OQ^2 - QM^2 = OQ^2 - SQ^2$$
$$= OY^2 - SY^2,$$

and OM'^2 may be shewn to have the same value

And since OM and OM' are thus equal, the line OV parallel to the axis bisects QQ', that is to say, it bisects every chord which is at right angles to OS

It is hence evident that every straight line parallel to the axis of a parabola is a diameter (Def. 7) of the curve, and that *all diameters of a parabola are parallel to the axis and to one another.*

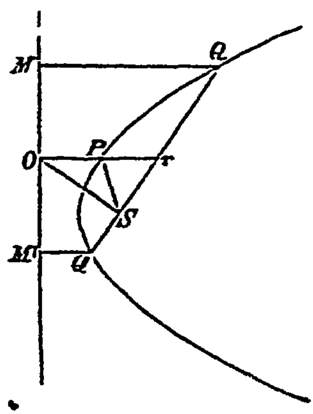
PROPOSITION III.

21 *The parameter of any diameter of a parabola is equal to four times the focal distance of its extremity*

Let a diameter meet its parameter QSQ' (Def. Art 17) in v, the curve in P, and the directrix in O, so that vSO is a right angle [Prop II.

Let fall perpendiculars QM and $Q'M'$ upon the directrix. Then

$$QQ' = SQ + SQ' = QM + Q'M'$$
$$= 2vO \qquad \text{[Lemma C}$$



And because vSO is a right angle, and $SP = PO$, therefore vO is a diameter of the circle round OSv, and P is its centre

Hence $QQ' = 2vO = 4SP$,

or the focal chord QQ' is equal to four times the focal distance of the extremity of the diameter of which it is the parameter. In particular, as we have already seen, the latus rectum or principal parameter is equal to $4AS$.

PROPOSITION IV

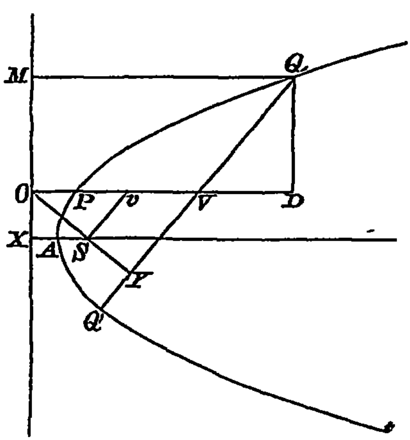
22 *The ordinate of any point on a parabola to any diameter is a mean proportional to the parameter of the diameter and the abscissa of the point*

Let Q be any point on a parabola, QV and PV its ordinate and abscissa to any diameter we have to shew that

$$QV^2 = 4SP.PV,$$

the parameter of the diameter through P being equal to $4SP$ [Prop III

Let the diameter meet its parameter in v and the directrix in O, and let OS, which is at right angles to Sv and QV (Prop II), meet QV in Y



Then it may be shewn that, if QD and QM be perpendiculars to the diameter and the directrix,

$$QD^2 = OM^2 = OY^2 - SY^2 \qquad \text{[Art 20}$$

But by similar triangles and parallels, QD is to QV as OY to OV and as SY to vV

Therefore $\qquad QV^2 = OV^2 - vV^2.$

And since P is the centre of the circle round OSv (Art 21), therefore OV is equal to $PV+SP$, and vV to $PV-SP$, and therefore from above,

$$QV^2=(PV+SP)^2-(PV-SP)^2,$$
$$=4SP\ PV; \qquad \text{[Lemma A}$$

or QV is a mean proportional to PV and $4SP$.

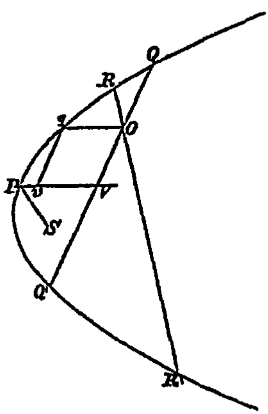
Corollary

Conversely, if the ordinate and abscissa of a point Q be so related that $QV^2=4SP\ PV$, where SP is any constant length, the locus of Q will be a parabola With centre P and radius equal to SP draw a circle cutting PV in O and v, and the parallel to QV through v in S Then, reversing the steps of the proposition, we have $QV^2=OV^2-vV^2$, and $OM^2=OY^2-SY^2$, and $SQ=QM$, or the locus of Q is a parabola whose focus and directrix are S and OM

PROPOSITION XV.

23 *The rectangles contained by the segments of any two intersecting chords are as the parameters of the diameters which bisect them*

Take any two chords QQ' and RR' intersecting in a point O, within or without the parabola, and let the diameter through O meet the parabola in q.



Bisect QQ' in V, let the diameter through V meet the curve in P, and draw the ordinate qv to that diameter Then, taking the case in which O lies within the parabola,

$$QO\ OQ' = QV^2 - OV^2, \qquad \text{[Euc II 5, Cor.}$$
$$= QV^2 - qv^2,$$
$$= 4SP\ PV - 4SP\ Pv, \qquad \text{[Prop IV}$$

and therefore, since $PV - Pv$ is equal to vV or qO,

$$QO\ OQ' = 4SP\ qO$$

Similarly $$RO\ OR' = 4SP'\ qO,$$

if P' be the extremity of the diameter which bisects RR', and the same may be proved for the case in which O lies without the parabola

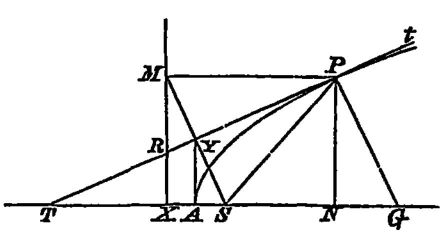
Therefore the rectangles $QO\ OQ'$ and $RO\ OR'$ are as $4SP$ to $4SP'$, or as the parameters of the diameters which bisect QQ' and RR' (Prop III), in accordance with what was proved for all conics without distinction in Art 16

Tangent-Properties.

PROPOSITION ~~VI~~ XI

24 *The tangent to a parabola at any point is the bisector of the angle which the focal distance of the point makes with the diameter produced*

Let the tangent at P meet the directrix in R, and let the diameter be produced through P without the curve to meet the directrix in M



Then PSR is a right angle (Art 7), and $SP = PM$, and PR is common to the right-angled triangles PSR and MPR

Therefore their angles at P are equal, or the tangent PR is the bisector of the angle SPM

It is further evident that RP produced makes equal angles with SP and PM, and that if the tangent meet the axis in T, the angles SPT and STP are equal, or the tangent together with SP and the axis determines an isosceles triangle.

Corollary 1

If PN be the principal ordinate of P, it follows that

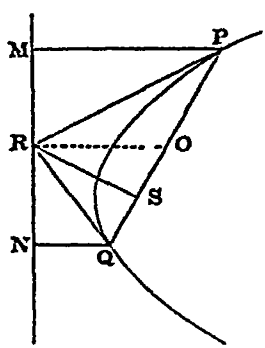
$$ST = SP = NX = AN + AS,$$

and therefore $\qquad AN = ST - AS = AT,$

or *the subtangent is double of the abscissa* [Def Art 17.

Corollary 2

If PQ be any focal chord and MN its projection upon the directrix, and if SR be drawn at right angles to PQ to meet the directrix, it is evident from above that the bisector of the



angle SRM is the tangent at P, and that the bisector of SRN is the tangent at Q Hence

$$\angle PRQ = \tfrac{1}{2}SRM + \tfrac{1}{2}SRN = \text{a right angle},$$

or the tangents at the extremities of any focal chord meet *at right angles* upon the directrix, and conversely

PROPOSITION VII

Y-HY

25 *The normal at any point makes equal angles with the focal distance and the axis, and it bisects the interior angle between the focal distance and the diameter through the point*

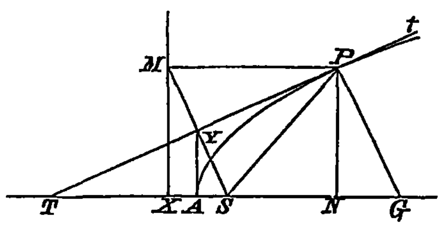
The normal at any point P is at right angles to the tangent

Let them meet the axis in G, T respectively, and let I be an internal point on the diameter through P.

Then the angles SPG, SGP are complementary to SPT, STP respectively And these are equal (Art 24)

Therefore $\angle SPG = SGP = GPI$,

or the normal makes equal angles with SP, SG and bisects the angle SPI.



Corollary

Hence, if PN be a principal ordinate,

$$SG = SP = NX,$$

or, subtracting SN from SG and from NX, the remainders NG, SX are equal, or *the subnormal is equal to the semi-latus rectum*

PROPOSITION VIII.

26 *The foot of the focal perpendicular to any tangent lies on the tangent at the vertex, and the square of the focal perpendicular to any tangent varies as the focal distance of its point of contact*

Let the tangent at P meet the axis in T, and let PN be the ordinate of P

(i) Draw the tangent at the vertex A, and let it meet PT in Y, then will SY be perpendicular to PT.

For since A is the middle point of NT (Art 24, Cor 1), therefore by parallels PN and AY (Art 3), Y is the middle point of PT; and therefore the triangles SPY and STY, having their sides SP and PY equal to ST and TY respectively, and the side SY common, have their angles at Y equal

Therefore SY is at right angles to PT, and conversely the foot of the focal perpendicular SY upon the tangent at P lies on the tangent at A.

(ii) Moreover, since the tangents YA and YP to the parabola subtend equal angles at S (Art 9), the right-angled triangles SAY and SYP are similar, so that

$$SA \, . \, SY = SY \, . \, SP,$$

or

$$SY^2 = SA \, . \, SP,$$

and SY^2 varies as SP, as was to be proved

PROPOSITION IX. XIX

27 *The exterior angle between any two tangents to a parabola is equal to the angle which either of them subtends at the focus*

Let the tangents at P and Q intersect in T, and let them meet the axis in p and q Take any point O in pS produced, and produce TS to any point t

Then by Euclid I 32, and because SPp is an isosceles triangle (Prop VI),

$$\angle PSO = SPp + SpP = 2SpP,$$

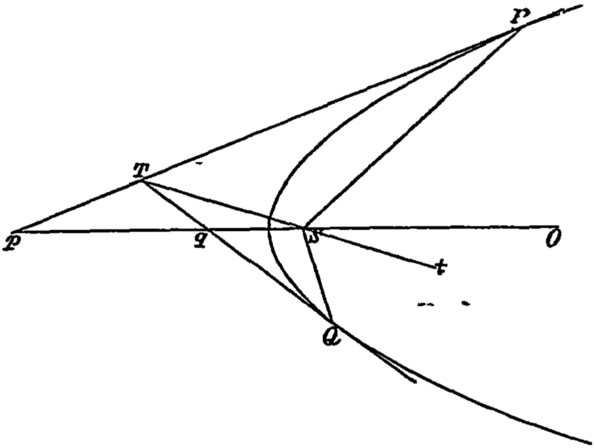
and similarly $\quad \angle QSO = 2SqQ$

Hence by addition (taking the case in which P and Q lie on opposite sides of the axis),

$$\angle PSQ = 2\,(SpP + SqQ) = 2\,(SpP + pqT),$$
$$= 2PTQ$$

Therefore $\angle PTQ = \frac{1}{2}PSQ = PSt = QSt$, [Art 9.

and therefore the exterior angle pTq between the tangents is equal to PST or QST, as was to be proved



If P and Q lie on the same side of the axis, then by subtraction,

$$\angle PSQ = 2\,(SqQ \sim SpP) = 2pTq,$$

or

$$\angle pTq = PST = QST.$$

PROPOSITION X

28 *The triangles described with any two tangents to a parabola as bases and the focus as vertex are similar to one another and to the triangle made by the two tangents and the axis or any triad of parallels thereto*

For in the same figure the triangles SPT and Tpq have their angles at S and T equal (Prop IX) and their angles at P and p equal (Prop VI), and are therefore similar In like manner the triangle STQ is similar to Tpq, and therefore to SPT

Hence $$SP\ \ ST = ST\ \ SQ,$$

or *the focal distance of the point of concourse of any two tangents to a parabola is a mean proportional to the focal distances of their points of contact*

Also, because the angle STQ is equal to PpS, *any two tangents to a parabola are equally inclined to the focal distance of their point of concourse and to the axis respectively* This may be remembered in the form

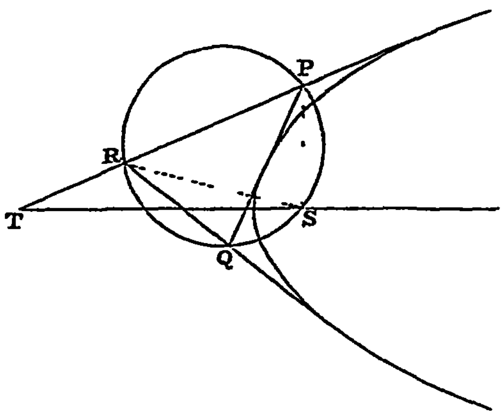
$$\angle STQ = HTP,$$

if H be an internal point on the diameter through T

PROPOSITION XI

29 *The circumscribed circle of the triangle formed by any three tangents to a parabola passes through the focus*

Let PQR be any triangle whose three sides touch a parabola, and let any one of its sides, as PR, meet the axis in T.



Then by Prop X, considering the two tangents which meet in R,

$$\angle SRQ = STP,$$

and next considering the two tangents which meet in P,

$$\angle SPQ = STP$$

Therefore the angles SPQ and SRQ are equal, and the points $PQRS$ lie on a circle, or in other words, the circle round PQR passes through the focus

Corollary

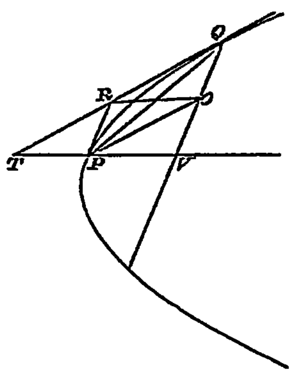
Four tangents to a parabola make four triangles whose circumcircles meet at the focus

PROPOSITION XII

30 *At any point of a parabola the subtangent to any diameter is double of the abscissa*

Let Q be any point on a parabola, QV and PV its ordinate and abscissa to any diameter, and let the tangent at Q meet that diameter in T we have to shew that TV is double of PV [Def. Art. 17.

Let the tangents at P and Q meet in R, and let PO be drawn parallel to RQ to meet QV in O, then the tangent at P, being a vanishing ordinate to the diameter PV (Art 13, Cor 1), is parallel to QV, and therefore $PRQO$ is a parallelogram, and its diagonal RO bisects the diagonal PQ*



But the line drawn from R to bisect the chord of contact of the two tangents from R is a diameter of the parabola (Art 13, Cor 2) therefore RO is a diameter, and is parallel to the diameter PV [Prop II

Hence by parallels,

$$PV = RO = PT,$$

or P bisects TV, and the subtangent TV is double of PV

* Any triangle whose base is parallel to the axis of a parabola has its other sides as the parallel tangents RP, RQ, these being as RP, RT

CHAPTER IV.

THE CENTRAL CONICS.

31 THE central conics are the ellipse and the hyperbola* It has been shewn (Art 14) that these are also bifocal, and symmetrical with respect to both axes

The *Abscissæ* or *Abscisses* of a point to any diameter of a central conic are the segments of the diameter made by the ordinate of the point The *Central Abscissa* which is also called absolutely the *Abscissa,* is the distance from the centre to the foot of the ordinate

The principal axis of a central conic is distinguished as the *Transverse Axis* The transverse and conjugate (Def 4) axes of the ellipse are also called its *Major* and *Minor* axes The term axis is sometimes used to denote the *finite* portion of either of these lines intercepted by the curve, but a special convention has to be made in case of the conjugate axis of the hyperbola, which does not meet the curve in real points [Art 33, Cor 1.

The major and minor auxiliary circles are the circles described upon the transverse and conjugate axes as diameters, but when one *Auxiliary Circle* only is spoken of the circle on the transverse axis is signified It is easily seen from the next article that this is identical with the eccentric circle of the centre of the conic, the ratio of its radius CA to CX being equal to the eccentricity

* We shall sometimes give proofs applicable to both curves, but with figures for the ellipse only In all such cases the student should draw the figures for the hyperbola also

Supplemental Chords are chords drawn from any point on the curve to opposite extremities of any diameter

The *Conjugate Parallelogram* is the figure formed by drawing parallels to each of two conjugate diameters through both extremities of the other

The locus of the point of concourse of a pair of tangents at right angles will be shewn to be a circle (Art 40), which we shall term the *Orthocycle**

32 *The segments of the transverse axis*

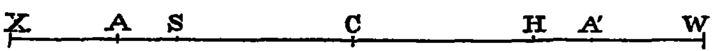
In any central conic the vertices A and A' divide SX so that

$$SA \quad AX = SA' \quad A'X,$$

therefore
$$SA \quad AX = SA \sim SA' \quad AX \sim A'X$$
$$= SA + SA' \quad AX + A'X,$$

or by Lemma B, if C be the centre of the conic,

$$SA \quad AX = 2CS \quad 2CA = 2CA \quad 2CX$$



Thus each of the ratios $CS \quad CA$ and $CA \quad CX$ is equal to the eccentricity, and

$$CS \quad CX = CA^2$$

The Ordinate.

PROPOSITION I.

33 *The square of the principal ordinate of any point on a central conic is in a constant ratio to the rectangle contained by its abscisses*

Let A and A' be the vertices, PN the ordinate of any point P on the conic, Z and Z' the points in which AP and $A'P$ or their prolongations meet the directrix

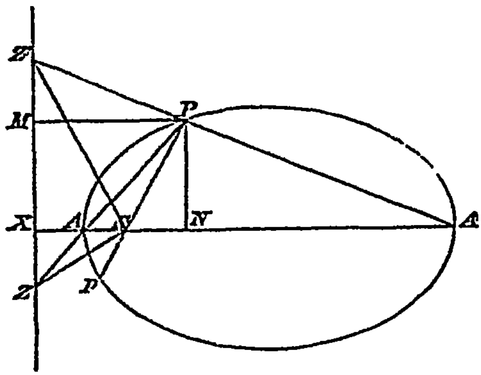
* It has also been named the *Director Circle*, since in the parabola it degenerates into the directrix and the line infinity

If PM be a perpendicular to the directrix, and therefore parallel to the axis,

$$SP : SA' = PM : A'X \qquad \text{[Def. 1.}$$

$$= PZ' : A'Z',$$

therefore SZ' bisects the angle PSX, and in like manner (if p be taken in PS produced) SZ bisects the supplementary angle pSX



Hence the angle ZSZ' is a right angle, and

$$ZX \cdot Z'X = SX^2 = \text{a constant}$$

Moreover, since PN is parallel to the directrix,

$$PN : AN = ZX : AX,$$

and
$$PN : A'N = Z'X : A'X$$

Hence $$PN^2 : AN \cdot A'N = ZX \cdot Z'X : AX \cdot A'X$$

$$= SX^2 : AX . A'X,$$

or PN^2 is in a constant ratio to $AN \cdot A'N$.

Corollary

Let the length CB be determined by the proportion,

$$CB^2 : CA^2 = SX^2 : AX \cdot A'X$$

Then in the ellipse

$$PN^2 : AN . A'N = PN^2 : CA^2 - CN^2 = CB^2 : CA^2,$$

where CB is equal to the semi-axis conjugate, with which PN coincides when N is taken at C In the hyperbola

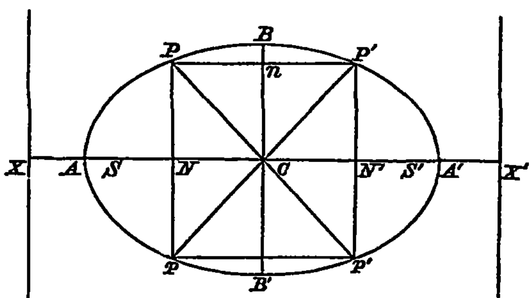
$$PN^2 \; AN \, A'N = PN^2 \; CN^2 - CA^2 = CB^2 \; CA^2,$$

where we may agree to call CB *the length of the semi-axis conjugate,* although (Art 2) this axis does not meet the curve in real points The results of this corollary may also be written,

$$CB^2 \pm PN^2 \; CN^2 = CB^2 \; CA^2,$$

the positive sign being taken in the hyperbola and the negative in the ellipse

If Pn be an ordinate to the conjugate axis, it is easily deduced that in the ellipse,



$$Pn^2 \; CB^2 - Cn^2 = CA^2 \; CB^2,$$

or $$CA^2 - Pn^2 \, . \, Cn^2 = CA^2 \; CB^2,$$

and in the hyperbola,

$$Pn^2 \; CB^2 + Cn^2 = CA^2 \; CB^2.$$ [Fig Art 49

PROPOSITION II

34 *The semi-axis conjugate is a mean proportional to the segments of the transverse axis made by either focus, and the latus rectum is a third proportional to the transverse and conjugate axes*

Since $CS \; CA = SA \; AX = SA' \; A'X,$ [Art 32

therefore $CS + CA \; CA = SA + AX \; AX = SX \; AX$

and $\quad CS \sim CA \,.\, CA = SA' \sim A'X \quad A'X = SX \quad A'X$

Hence $\quad CS^2 \sim CA^2 \cdot CA^2 = SX^2 \quad AX.A'X$

$$= CB^2 \quad CA^2, \qquad \text{[Art. 33}$$

or $$CS^2 \sim CA^2 = AS \quad A'S = CB^2.$$

If LSL' be the semi-latus rectum, so that AS and $A'S$ are the abscissæ of L,

$$SL^2 \,.\, AS \quad A'S = CB^2 \quad CA^2, \qquad \text{[Prop I}$$

and therefore $\quad SL^2 \quad CB^2 = CB^2 \,.\, CA^2$

Therefore $SL \quad CA = CB^2$, and LL' is a third proportional to the transverse and conjugate axes

PROPOSITION III

35 *If the principal ordinates of all points on an ellipse be produced outwards in the ratio of the major to the minor axis, the points to which they are produced lie on the circumference of the auxiliary circle, and conversely*

Let P be any point on an ellipse, and let its principal ordinate PN be produced outwards to p in the ratio of CA to CB, so that [Fig p 64

$$PN^2 \,.\, pN^2 = CB^2 : CA^2 = PN^2 : AN.A'N \qquad \text{[Prop I}$$

Then, pN^2 being equal to $AN \quad A'N$, the locus of p is the circle on AA' as diameter, as was to be proved.

Conversely, if the ordinates pN, qM .. in the circle be cut in the ratio of minority $CB \quad CA$, the points of section P, Q . will all lie on an ellipse whose semi-axes are equal to CA and CB [Fig Art 61

In virtue of this relation we are able to deduce a whole class of properties of the ellipse from properties of the circle*

* The corresponding chords PQ and pq always meet on the axis Hence the tangents at P and p meet on the axis, viz. in a point T such that $CN \quad CT = Cp^2 = CA^2$ (Fig Art 45) Cf Art 68 Note that Art *47 (p 64) enables us to deduce all the projective properties of the ellipse (Prop I not excepted) from the circle.

by the method of Orthogonal Projection, which will be explained in Chapter VIII

Let the minor auxiliary circle cut any ordinate Pn (Fig p 46) to the minor axis in a point Q

Then by Prop I Cor, and by a property of the circle, Pn^2 is to $Bn \cdot nB'$ (or Qn^2) as CA^2 to CB^2

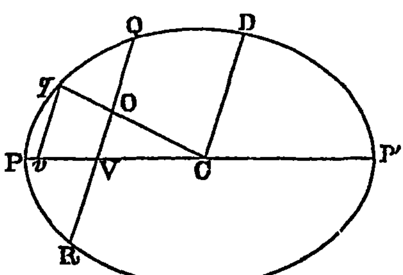
Therefore $\qquad Pn : Qn = CA : CB.$

Hence and by the preceding converse, if the ordinates to a diameter of a circle be cut in a given ratio of minority or majority, the point of section will lie on an ellipse having that diameter for major or minor axis

PROPOSITION IV.

36 *At any point on a central conic the square of the ordinate to any diameter is in a constant ratio to the product of the corresponding abscissæ*

Let QVR be any double ordinate of a given diameter PP' and PV and $P'V$ the corresponding abscissæ We have to shew that QV^2 (or RV^2) varies as $PV \cdot VP'$



This follows at once from Art 16, Cor. 1, where it is shewn that $QV \cdot VR$ (which is in this case equal to QV^2) is in a constant ratio to $PV \cdot VP'$, so long as the directions of QR and PP' continue unchanged.

Put $CD^2 : CP^2$ equal to this constant ratio, so that

$$QV^2 : PV \cdot VP' = CD^2 : CP^2$$

In the ellipse it is evident that CD is equal to the semi-diameter to which QV is parallel, and that

$$PV \cdot VP' = CP^2 - CV^2,$$

in all cases.

In the hyperbola also we may agree to call *CD the length of the semi-diameter conjugate to CP*, although it will be seen (in the section on conjugate diameters) that one and one only of every two conjugate diameters meets the curve in real points. Supposing CP to meet the hyperbola, then

$$QV^2 : CV^2 - CP^2 = CD^2 : CP^2,$$

or $$QV^2 + CD^2 : CV^2 = CD^2 : CP^2.$$

Hence also $$Qv^2 : Cv^2 + CD^2 = CP^2 : CD^2,$$

if Qv (equal to CV) be an ordinate to the diameter CD, which does *not* meet the curve.

Corollary.

Hence, and by Art. 16, Cor. 2, if FF' be the focal chord parallel to CD, and LL' be the latus rectum,

$$FF' : LL' = CD^2 : CB^2 = CD^2 : \tfrac{1}{2}LL' \cdot CA \qquad \text{[Art. 34.}$$

Therefore $$FF' \cdot CA = 2CD^2,$$

or any focal chord is a third proportional to the transverse axis and the diameter parallel to the chord.

The Focal Distances.

PROPOSITION V.

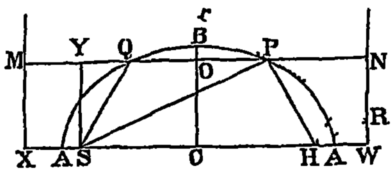
37. *The sum or difference of the focal distances of any point on a central conic is constant and equal to the transverse axis.*

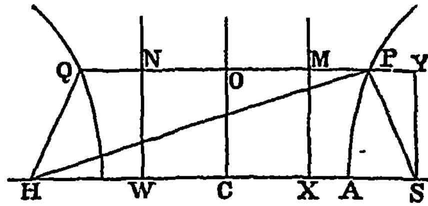
Let H be the second focus (Art. 14) and W the foot of the corresponding directrix. Through any point P on the curve draw a parallel to the axis to meet the directrices in M and N. Then

$$SP : PM = HP : PN$$

Therefore

$$SP \pm HP . PM \pm PN = SP \quad PM,$$





where $PM + PN$ is equal to WX in the ellipse, and $PM \sim PN$ is equal to WX in the hyperbola

Therefore $SP \pm HP$ is constant, the upper sign being taken for the ellipse, and the lower sign for the hyperbola

By making P coincide with A, it is easily seen that in either case the constant length is equal to the transverse axis, so that

$$SP \pm HP = AA'$$

Corollary

In the ellipse, if B be an extremity of the minor axis,

$$SB = HB = CA$$

The Tangent.

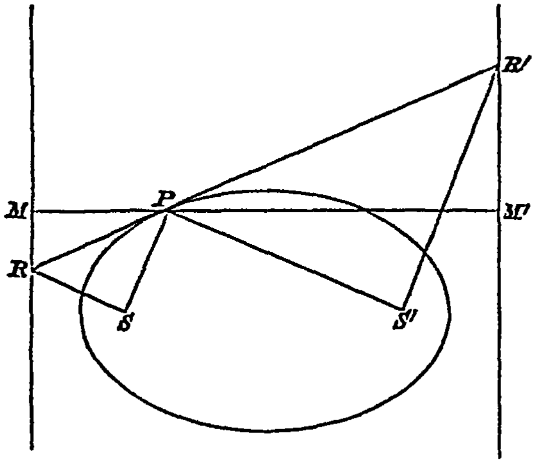
PROPOSITION VI.

38 *The tangent at any point of a central conic makes equal angles with the focal distances of the point*

Let the tangent at P meet the directrices in R and R', and let the parallel to the axis through P meet the directrices in M and M' Then, if S and S' be the corresponding foci,

$$SP \quad S'P = PM \quad PM' = PR \quad PR',$$

or the sides about the angles at P in the triangles PSR and $PS'R'$ are proportionals



Moreover the angles at S and S' in those triangles (being right angles) are equal.

Therefore the triangles are similar and their angles at P are equal (Euc VI 7), that is to say, the tangent at P makes equal angles with the focal distances SP and $S'P$, and conversely

In the ellipse the tangent at P falls without the angle SPS', and bisects the angle which $S'P$ makes with SP produced (Fig Art 39) In the hyperbola the tangent bisects the angle SPS' internally.

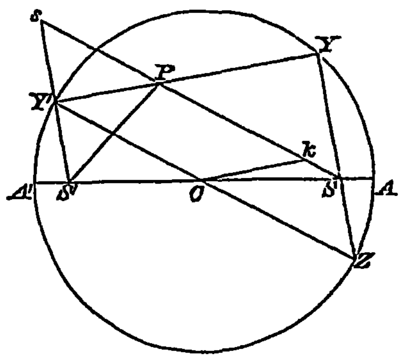
PROPOSITION VII

39 *The projections of the two foci upon any tangent to a central conic lie on the auxiliary circle*

Let S and S' be the foci Y and Y' the feet of the perpendiculars let fall from them upon the tangent at any point P Then will Y and Y' lie on the auxiliary circle

In the case of the ellipse, let SP produced meet $S'Y'$ in

s, then the tangent PY' bisects the angle $S'Ps$ (Prop VI) and $Y's$ is equal to $Y'S'$



And since Y' has been shewn to be the middle point of $S's$, and the centre C of the conic bisects SS', therefore CY' is parallel to Ss and equal to $\frac{1}{2}Ss$. That is to say,

$$CY' = \tfrac{1}{2}(SP + Ps) = \tfrac{1}{2}(SP + PS')$$
$$= CA, \qquad \text{[Prop V}$$

or Y' lies on the auxiliary circle And in like manner it may be shewn that Y lies on the auxiliary circle.

In the hyperbola it may be shewn in like manner that

$$CY = CY' = \tfrac{1}{2}(SP \sim S'P) = CA,$$

and thus that Y and Y' lie on the auxiliary circle

Corollary 1.

Conversely, if Y be any point on the auxiliary circle the straight line drawn from Y at right angles to SY will be a tangent to the conic It is hence evident that the extremities of any focal chord YSZ of the auxiliary circle are the projections of the focus S upon a pair of *parallel tangents* to the conic

Corollary 2.

If the diameter parallel to the tangent at P meet SP in k, then

$$Pk = CY' = CA.$$

PROPOSITION VIII

40 *The rectangle contained by the perpendiculars from either focus of a central conic upon any two parallel tangents, or by the perpendiculars from the two foci upon any tangent, is constant and equal to the square of the semi-axis conjugate*

Draw perpendiculars SY and SZ to any two parallel tangents Then since Y and Z lie on the auxiliary circle (Prop VII.) and YSZ is a straight line,

$$SY\ SZ = SA\ SA' = CB^2,$$ [Prop II

as was to be proved.

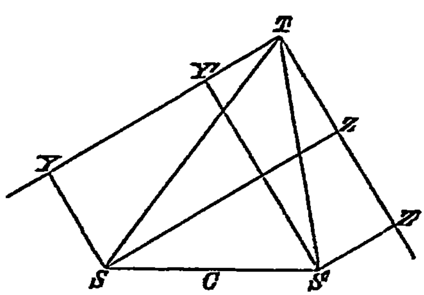
Again, with the construction of Art. 39, since $Y'YZ$ is a right angle $Y'Z$ is a diameter of the circle Hence the triangles CSZ and $CS'Y'$, having their angles at C equal and their sides CS, CZ equal to CS', CY' each to each, are equal in all respects

Therefore $$SY\ S'Y' = SY.SZ = CB^2,$$

as was to be proved.

Corollary 1

If tangents be drawn to an ellipse from any point T, and if Y, Y' and Z, Z' be the projections of the foci upon them, then since $SY\ S'Y' = SZ\ S'Z'$, it is easily seen that the



angle STY must be equal to $S'TZ$, for according as the angle STY is equal to or greater or less than $S'TZ$, the angle $S'TY'$ is equal to or greater or less than STZ', and the ratio of $SY\ S'Y'$ to $ST\ S'T$ is equal to or greater or less than the ratio of $SZ\ S'Z'$ to $ST\ S'T$ [See also Art 98

Corollary 2

In the preceding figure let the tangents be supposed to be at right angles Draw the auxiliary circle, viz through the points $YY'ZZ'$, and draw TO touching it in O Then

$$TO^2 = TY\ TY' = SZ\ S'Z' = CB^2,$$

and $$CT^2 = CO^2 + TO^2 = CA^2 + CB^2$$

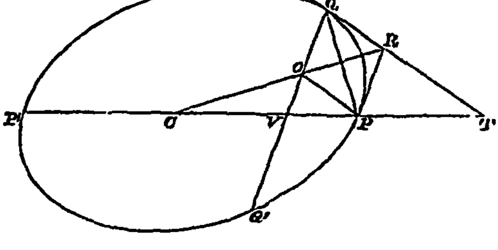
That is to say, *the locus of the point of concourse of a pair of tangents at right angles is a circle,* viz the *Orthocycle** In the hyperbola it will be found that the radius of the orthocycle is equal to $\sqrt{(CA^2 - CB^2)}$ Consequently no real tangents at right angles can be drawn to a hyperbola in which CA is less than CB

PROPOSITION IX

41 *A variable tangent to a central conic meets any fixed diameter at a distance from the centre which varies inversely as the abscissa of its point of contact to that diameter*

First Case

Let PCP' be a given diameter which *meets* the curve in real points P and P' Let the tangent at any point Q meet



* See also note, page 44

that diameter in T, and let QV be the ordinate of Q. Then we shall shew that

$$CV.CT=CP^2.$$

Let the tangent at P, which is parallel to QV, meet QT in R, and let PO be drawn parallel to RQ to meet QV in O. Then $OPRQ$ is a parallelogram, and its diagonal RO bisects PQ.

But RO, since it bisects the chord of contact of the tangents from R, is a diameter and passes through the centre C of the conic. [Art. 13, Cor. 2.

Therefore by parallels,

$$CV : CP = CO : CR = CP : CT,$$

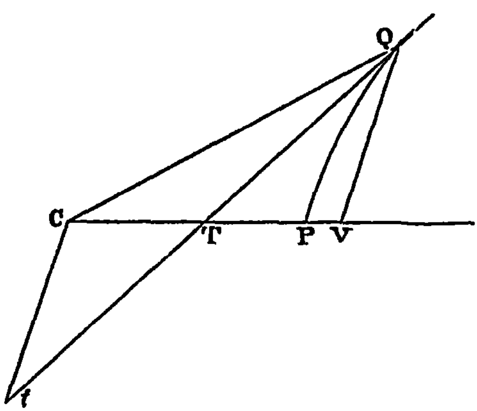
or $CV . CT$ is equal to CP^2, as was to be proved.

Second Case.

Let CP and CD be given conjugate semi-diameters of a hyperbola, whereof the former meets and the latter does *not meet* the curve. [Art. 36.

Draw QV and QT as in the first case, and produce QT to meet CD in t. Then by parallels,

$$Ct : CT = QV : VT,$$



or
$$QV \; Ct \quad CV \; CT = QV^2 \quad CV \; VT$$
$$= QV^2 \quad CV^2 - CV \; CT$$

Hence by the first case and by Art 36,
$$QV \; Ct \quad CP^2 = QV^2 \quad CV^2 - CP^2$$
$$= CD^2 \quad CP^2,$$
or
$$Cv \quad Ct = QV \; Ct = CD^2,$$
if Cv be the abscissa of Q to the diameter CD

Corollary

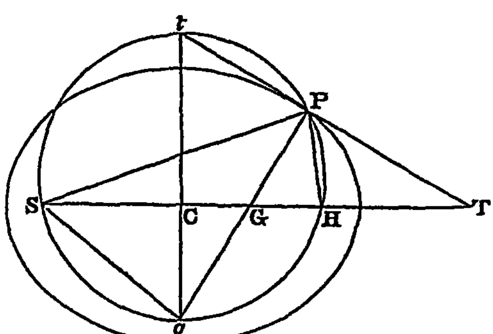
If the tangent at any point P meet the transverse and conjugate axes in T and t, and if PN and Pn be ordinates to those axes, then
$$CN . CT = CA^2, \text{ and } Cn \quad Ct = CB^2.$$

The Normal.

PROPOSITION X

42 *The normal at any point bisects the angle between the focal distances of the point, internally in the case of the ellipse and externally in the case of the hyperbola*

Let S and H be the foci, PG the normal at any point P, and TPt the tangent at that point



Then in the ellipse, PG being at right angles to the tangent,

$$\angle SPG + SPt = HPG + HPT.$$

And it has been shewn that the angles SPt, HPT are equal and that the tangent falls without the angle SPH (Art 38)

Therefore $\angle SPG = HPG,$

or the normal bisects SPH *internally*.

In the hyperbola it may be shewn in like manner that the normal at P bisects the angle SPH *externally*

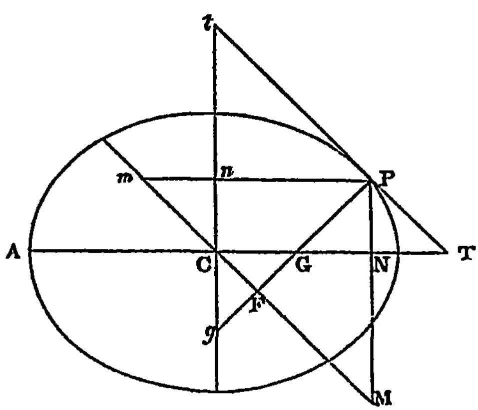
Corollary

If the circle round SPH meet the conjugate axis in g and t, it is evident that the arcs Sg, Hg and the arcs St, Ht are equal Hence Pg and Pt are the two bisectors of the angle SPH, *and the circle round SPH passes through the points in which the tangent and normal at P meet the conjugate axis*

PROPOSITION XI

43 *At any point of a central conic the normal, terminated by either axis, varies inversely as the central perpendicular upon the tangent*

Let the tangent at any point P meet the transverse and



conjugate axes in T and t, and let the normal meet them in G and g

Let PN and Pn be ordinates to those axes, and let PN and Pn or their prolongations meet the diameter parallel to the tangent at P in M and m, and let the normal meet that diameter in F

Then, the angles at N and F being right angles, a circle goes round $FGNM$, and therefore

$$PG\ \ PF = PN\ \ PM = Cn\ \ Ct$$

$$= CB^2 \qquad \text{[Art 41, Cor}$$

In like manner, the angles at n and F being right angles,

$$Pg\ \ PF = Pn\ \ Pm = CN\ \ CT$$

$$= CA^2$$

Therefore PG and Pg vary inversely as PF, or as the central perpendicular upon the tangent at P

Corollary

Hence

$$NG\ \ CN = NG\ \ Pn = PG\ \ Pg = CB^2\ \ CA^2,$$

or *the subnormal varies as the abscissa.* [Def Art. 17

Conjugate Diameters.

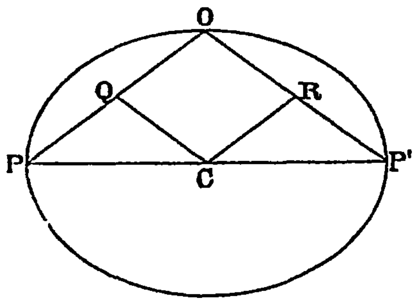
PROPOSITION XII.

44 *Supplemental chords are parallel to conjugate diameters, and conversely*

Let PCP' be any diameter and O any point on the curve. Then will the supplemental chords OP and OP' (Def Art 31) be parallel to a pair of conjugate diameters

For if Q be the middle point of OP, and R the middle point of OP', the line CQ which bisects two sides of the

triangle OPP' is parallel to the third side $P'O$, and in like manner CR is parallel to PO.



Therefore the diameters CQ and CR are conjugate, since each bisects one chord (and therefore all chords) parallel to the other [Art 13

Conversely, from any given point O on the curve or from the extremities of any given diameter PP' there can be drawn a pair of supplemental chords OP and OP' parallel to any assumed pair of conjugate diameters

In the hyperbola it is evident that of every two supplemental chords one lies within and the other without the curve, and hence that *one and one only of every two conjugate diameters meets the hyperbola* We shall in consequence have occasion to give separate proofs of some of the properties of conjugate diameters for the special case of the hyperbola

Corollary

This proposition determines the relation between the directions of any two conjugate diameters For in Art 33, where AP and $A'P$ may be parallel to any two such diameters,

$$PN^2 . AN\ A'N = CB^2\ CA^2,$$

and therefore if the ratio of PN to AN (or the direction of one of the diameters) be given, the ratio of PN to $A'N$ (or

the direction of the conjugate diameter) is known. Conversely any two diameters whose directions are thus related will be conjugate provided that they lie in adjacent quadrants in the case of the ellipse (Fig. Art. 45), or in the same quadrant or opposite quadrants in the case of the hyperbola. If CP and CD be conjugate radii, CN and CR their projections upon the axis, it is easily deduced that

$$PN.DR : CN.CR = CB^2 : CA^2.$$

PROPOSITION XIII.

45. *The sum of the squares of any two conjugate diameters is constant in the ellipse, and the difference of their squares is constant in the hyperbola.* [See also Art. 66.

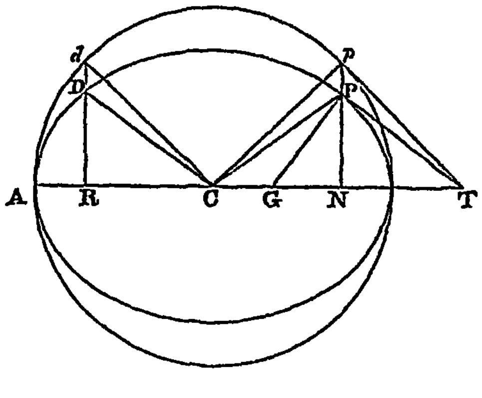
(1) Let CP and CD be conjugate semi-diameters of an ellipse, PN and DR principal ordinates.

Then because P, D are on the curve (Art. 33),

$$PN^2 : CA^2 - CN^2 = DR^2 : CA^2 - CR^2 = CB^2 : CA^2,$$

and because CP, CD are conjugate,

$$PN.DR : CN.CR = CB^2 : CA^2. \qquad \text{[Art. 44, Cor.}$$



Hence CA^2 must be equal to $CN^2 + CR^2$. For this makes each of the ratios $PN : CR$ and $DR : CN$ equal to

$CB \cdot CA$, and the ratio $PN \ DR \cdot CN \ CR$ compounded of them equal to $CB^2 \ CA^2$ Whereas, if CA^2 were not equal to $CN^2 + CR^2$, the former ratios would be both greater or both less than $CB \ CA$, and the ratio compounded of them greater or less than $CB^2 \ CA^2$.

Hence, because CP and CD are conjugate,

$$PN \ \ CR = DR \ \ CN = CB \ \ CA,$$

and $$CN^2 + CR^2 = CA^2,$$

and therefore $$PN^2 + DR^2 = CB^2$$

By addition, $$CP^2 + CD^2 = CA^2 + CB^2,$$

or *the sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axes of the ellipse*

(ii) In the hyperbola*, as will be proved in Arts 52, 89,

$$PN . CR = DR \ \ CN = CB \ \ CA,$$

and $$CP^2 \sim CD^2 = CA^2 \sim CB^2.$$

Corollary 1

If P be any point on an ellipse, $(SP + S'P)^2 = 4CA^2$, and $SP^2 + S'P^2 = 2CS^2 + 2CP^2$ (Lemma D). Therefore, subtracting and dividing by 2,

$$SP \ \ S'P = 2CA^2 - CS^2 - CP^2 = CA^2 + CB^2 - CP^2$$
$$= CD^2$$

The same may be proved for the hyperbola in like manner, or as in Art. 86

Corollary 2

If the normal at P meet the transverse axis in G, then by similar triangles (CD being at right angles to the normal)

$$PG \ \ CD = PN \ \ CR = CB \ \ CA.$$

* No separate proof would be wanted for the hyperbola if we might use imaginary points (Schol p 70)

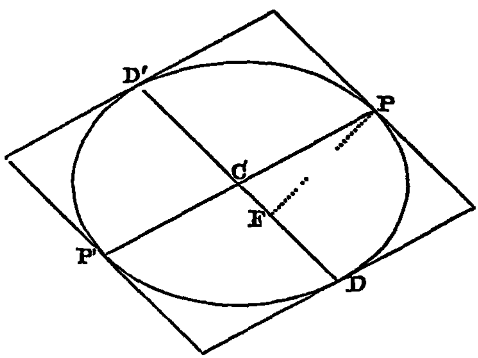
Similarly $\qquad Pg\ \ CD = CA\ \ CB,$

where Pg is the normal terminated by the conjugate axis

PROPOSITION XIV

46 *The conjugate parallelogram is of constant area and equal to the rectangle contained by the axes*

Let PCP' and DCD' be a pair of conjugate diameters, and let a conjugate parallelogram be constructed by drawing parallels to each of them through the extremities of the other [Def Art 31.



Let the normal at P meet DD' in F and the transverse axis in G Then since

$$PG\ \ CD = CB\ \ CA, \qquad \text{[Art 45, Cor 2}$$

therefore

$$PF\ PG\ \ PF\ CD = CB^2\ \ CA . CB.$$

Hence, the antecedents being equal by Prop XI,

$$PF\ CD = CA\ \ CB,$$

and the conjugate parallelogram is equal to $4 PF\ CD$ or $AA'\ BB'$

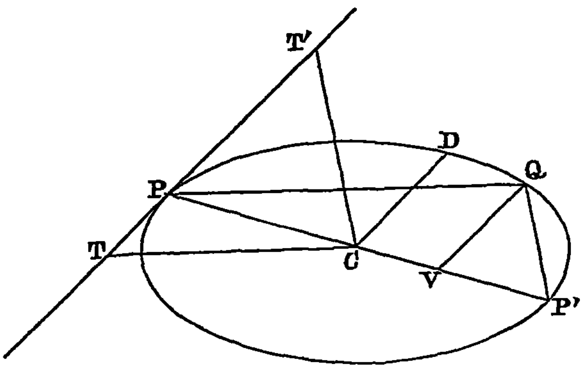
In the ellipse the parallelogram described as above completely envelopes the curve in the hyperbola two of its sides only touch the curve (Fig Art 50)

PROPOSITION XV

47 *The intercepts on any tangent between the curve and any two conjugate diameters contain a rectangle equal to the square of the semi-diameter parallel to the tangent*

Draw the tangent at any point P, the diameter PCP' and the conjugate semi-diameter CD

Let any second pair of conjugate diameters meet the tangent at P in T and T', and draw the supplemental chords QP and $P'Q$ parallel to CT and CT' [Art 44



Then by similar triangles, if QV be an ordinate to PP',

$$PT : CP = QV . PV,$$

and $$PT' : CP = QV : P'V$$

Hence $$PT.PT' : CP^2 = QV^2 : PV.P'V$$

$$= CD^2 : CP^2,$$ [Art. 36.

or $PT.PT'$ is equal to CD^2, as was to be proved

Corollary

In the hyperbola the points T and t in which any two conjugate diameters meet the tangent at Q lie on the same side of Q, as in the second figure of Art 41, and the two diameters may be supposed to coalesce In this case the intercept QT becomes equal to the parallel semi-diameter

CD A self-conjugate diameter is a tangent at infinity, since it is parallel to the tangents at its own extremities, which are therefore points at infinity, and a tangent at infinity must pass through the centre [Art 41

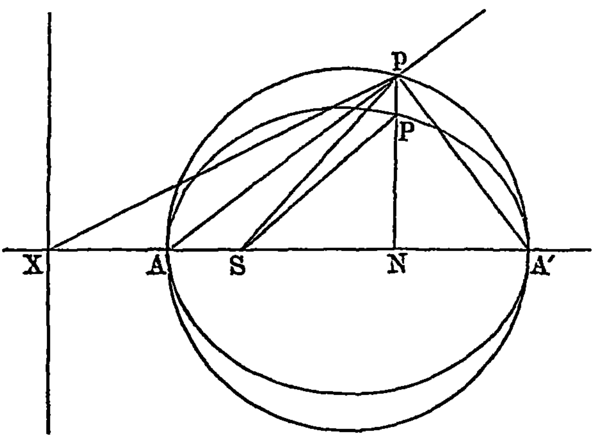
*47 *A new way of beginning the ellipse*

At a point P on the ellipse whose ordinate is PN

$$SP \quad NX = SA \quad AX = \text{the eccentricity} \qquad [\text{Art } 1$$

Produce NP to p, so that (A' being the further vertex)

$$Sp \quad pX = SA \quad AX = SA' \quad A'X,$$



then pA bisects the angle SpX, and pA' bisects its supplement, and ApA' is a right angle, or p lies on the *Auxiliary Circle.*

Hence by Euclid I 47, if e denote the eccentricity,

$$pN^2 - PN^2 = Sp^2 - SP^2 = e^2 (pX^2 - NX^2) = e^2 \quad pN^2,$$

and therefore PN is in a constant ratio to pN

Hence (1) an ellipse of any eccentricity can be described by cutting the ordinates to a diameter of a circle in a constant ratio (Fig p 47), and (2) PN^2 varies as $AN \quad NA'$, this being equal to pN^2 in the circle, and (3), if PN^2 thus vary as $AN . NA'$, it may be shewn conversely that $SP \quad NX$ is a constant ratio

CHAPTER V

THE ASYMPTOTES

48 IF a straight line and a curve, being produced, continually approach one another but never actually meet until they are produced to infinity, the straight line is said to be an *Asymptote* of the curve It will be seen that every hyperbola has *two asymptotes*

If CE and CP be two diameters of a hyperbola lying in the same quadrant, and EPN be an ordinate to the transverse axis, the diameters will be conjugate provided that

$$EN\ PN\ \ CN^2 = CB^2\ \ CA^2 *$$

If CP be made to coalesce with CE, it follows that

$$EN\ \ CN = CB\ \ CA,$$

a relation which determines the position of the *self-conjugate* diameter CE The curve has also a second self-conjugate diameter CE' making the same angle as CE with the axis

Since one of every two conjugate diameters meets and the other does not meet the hyperbola (Art 44), a diameter which is self-conjugate should meet and yet not meet the curve The self-conjugate diameters accordingly coincide with the asymptotes, which meet the curve at infinity and do not meet it at any assignable distance from the centre See also Art 47, Cor.

For the sake of uniformity of enunciation we shall at the outset speak of the lines CE and CE' as the asymptotes, and shall then shew conversely that they possess the property from which the term asymptote is derived [Prop I Cor

* In Art 44, Cor, let CD meet the ordinate of P in E

It is to be noticed that the asymptotes are the diagonals of the rectangle formed by drawing parallels to each axis through both extremities of the other. On the tangent at A take a length Aa equal to CB, then will Ca be equal to CS (Art 34), and $Ca : CA$ to the eccentricity. Also it may be easily proved that the feet of the focal perpendiculars upon the asymptotes lie on the auxiliary circle, and that each perpendicular is equal to Aa or CB and touches the eccentric circles of all points on the same asymptote.

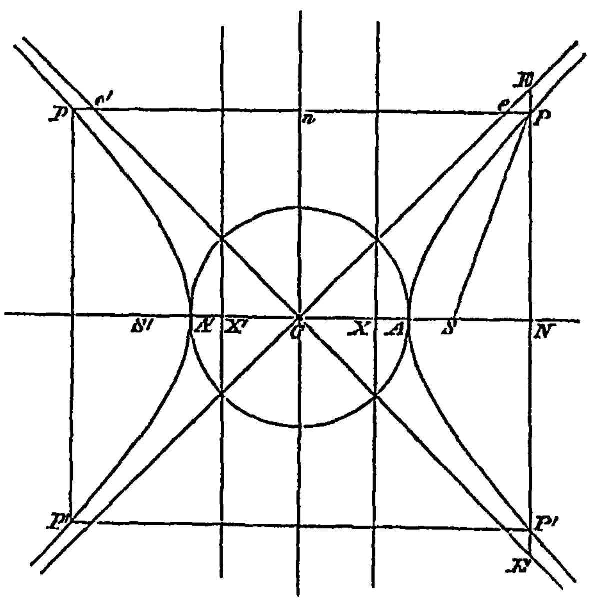
[See also Arts 82, 84.

PROPOSITION I

49. *The rectangle contained by the distances of any point on a hyperbola from its two asymptotes is of constant magnitude.*

(1) If the principal ordinate of any point P on the hyperbola meet the axis in N and the asymptotes in E and E', then

$$PN^2 + CB^2 : CN^2 = CB^2 : CA^2 \qquad \text{[Art 33}$$
$$= EN^2 : CN^2, \qquad \text{[Art. 48.}$$



and therefore $PN^2 + CB^2$ is equal to EN^2 and

$$PE \,.\, PE' = EN^2 - PN^2 = CB^2.$$

(ii) In like manner, if a parallel to AA' meet the curve in P, p and the asymptotes in e, e', then

$$Pe \, . \, Pe' = eP \, . \, ep = CA^2.$$

(iii) Next let Pe be drawn in any specified direction to meet CE. Then the triangle PEe is given in species and Pe varies as PE, and therefore $Pe \; PE'$ is constant

In like manner, if Pe' be drawn in any specified direction* to meet CE' the length Pe' varies as PE', and therefore $Pe \, . \, Pe'$ is constant

Taking for example the case in which PO the distance of P from one asymptote is measured parallel to the other (Fig Art 52), and CO is therefore equal to the distance of P from the latter asymptote measured parallel to the former, we have $PO \; CO$ constant, and it may be shewn by making P coincide with the vertex that

$$PO \, . \, CO = \tfrac{1}{2} CS \, . \, \tfrac{1}{2} CS = \tfrac{1}{4} (CA^2 + CB^2) \qquad \text{[Art 48}$$

Corollary

Since $PE \; PE'$ is constant and PE' continually increases with CN, therefore PE at the same time continually decreases Hence CE and the curve continually approach one another but never actually meet until produced indefinitely†

PROPOSITION II.

50 *The intercepts on any tangent to a hyperbola between the curve and its asymptotes are equal to one another and to the parallel semi-diameter; and the opposite intercepts on any chord between the curve and its asymptotes are equal to one another.*

* In this construction Pe and Pe' are not required to be in the same straight line

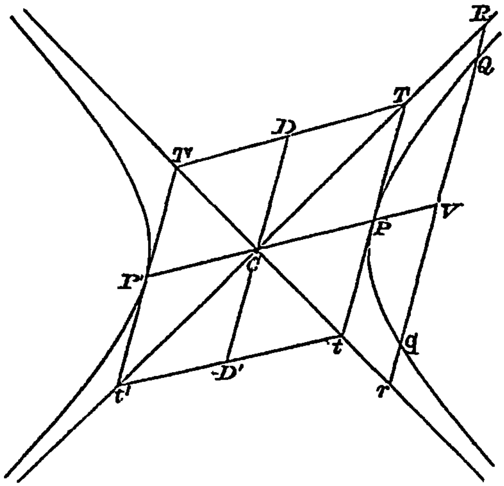
† The two branches of a hyperbola constitute one complete and continuous curve, which may be regarded as the locus of a point moving as follows First let the point trace the quadrant AP and recede to infinity in the direction CE then, travelling in the same direction, it reappears at the opposite extremity of the asymptote (Lemma H), traces the branch $p'A'p$, and finally the quadrant $P'A$. This appears plainly when the hyperbola is traced with the help of the eccentric circle, the point p (Art 6) being supposed to move continuously round the circle, governing the motion of the point P which traces the conic

(i) Let the tangent at P be parallel to the semi-diameter CD, and let it meet the asymptotes in T and t

Through T draw a chord aa' parallel to the axis Then

$$TP^2 : CD^2 = Ta \cdot Ta' : CA^2 \qquad \text{[Art 16}$$

$$= CA^2 : CA^2, \qquad \text{[Prop I (ii)}$$



or TP is equal to CD, and in like manner tP is equal to CD Therefore

$$PT = CD = Pt,$$

as was to be proved.

It readily follows that every two parallel tangents as TPt and $T'P't'$ terminated by the asymptotes are sides of a conjugate parallelogram (Def Art 31), and conversely that every conjugate parallelogram has its diagonals coincident with the asymptotes

(ii) Next let any chord Qq parallel to the tangent TPt be produced to meet the asymptotes in R and r Then the diameter CP bisects Rr, and it also bisects the chord Qq

[Art 13, Cor 1

Therefore the opposite intercepts QR and qr are equal and the opposite intercepts Qr and qR are equal, as was to be proved And the same may be shewn in like manner if the chord be supposed parallel to the diameter PP' which meets the curve in real points

PROPOSITION III

51 *A chord of a hyperbola being cut by either asymptote, to determine the rectangle contained by its segments*

In the preceding figure, through R draw a chord aa' parallel to the axis Then

$$RQ \;\; Rq \quad CD^2 = Ra \;\; Ra' \quad CA^2 \qquad \text{[Art 16}$$

$$= CA^2 \quad CA^2, \qquad \text{[Prop I (ii)}$$

or, if Qq be any chord, $RQ \, . \, Rq$ is equal to CD^2 See also Art 16, Cor 4

In like manner $rq \, . \, rQ$ is equal to CD^2

Hence Prop II may be deduced, and it follows that

$$RQ \quad Qr = rq \quad qR = CD^2$$

PROPOSITION IV

52 *The difference of the squares of any two conjugate semi-diameters of a hyperbola is equal to the difference of the squares of the semi-axes, and the triangle contained by the asymptotes and any tangent is equal to the rectangle contained by the semi-axes*

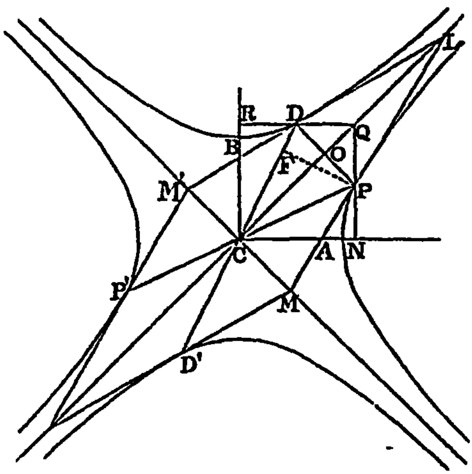
(1) Let the tangent at P meet the asymptotes in L and M. Draw PY perpendicular to CL, and bisect CL in O Then, P being the middle point of LM (Prop II), OP is parallel to the asymptotes CM, and the triangle OYP is given in species and OY varies as OP

Also, by Euc I 47 and Lemma A,

$$CP^2 \sim PL^2 = CY^2 \sim LY^2 = 4CO \cdot OY,$$

where OY varies as OP, and therefore inversely as CO [Prop I

Therefore $CP^2 \sim CD^2$ or $CP^2 \sim PL^2$ has a constant magnitude, which may be shewn by taking P at the vertex to be equal to $CA^2 \sim CB^2$



(ii) The rectangle $CL.CM$ is equal to $2CO.2PO$ or CS^2 (Art 49) Therefore the triangle LCM is of constant area, and it may be shewn by taking P at the vertex to be equal to $CA \cdot CB$

The *Conjugate Parallelogram*, which is four times the triangle CTt (Art 50), is therefore equal to $AA' \cdot BB'$.

Scholium

Although, in accordance with an ancient convention, we have assigned specific lengths to those diameters of the hyperbola which do not meet the curve (Art 36), such diameters have notwithstanding no real extremities or magnitudes, and it is therefore

impossible to treat the ellipse and the hyperbola altogether similarly so long as we restrict ourselves to the conception of real points. It may seem that in the corollary of Art 33 the two curves have been treated similarly but this is not the case In the ellipse, regarded as the locus of a point P whose ordinate and abscissa are connected by the relation,

$$PN^2 . CA^2 - CN^2 = CB^2 \; CA^2,$$

we find (1) by making PN vanish that the transverse semi-axis is equal to CA, and (2) by making CN vanish that the conjugate semi-axis is equal to CB In the hyperbola, from the relation,

$$PN^2 \; CN^2 - CA^2 = CB^2 . CA^2,$$

we find (1) by making PN vanish that the transverse semi-axis is equal to CA, and (2) by making CN vanish that the square of the conjugate semi-axis is equal to the negative quantity $-CB^2$. In like manner, from the relation of Art 36,

$$QV^2 . CV^2 - CP^2 = CD^2 . CP^2,$$

we find by making CV vanish that the square of the semi-diameter conjugate to CP is equal to the negative quantity $-CD^2$ For $-CB^2$ write $C\beta^2$ and for $-CD^2$ write $C\delta^2$, then the hyperbola may be treated as a quasi-ellipse, in which

$$PN^2 \; CA^2 - CN^2 = C\beta^2 \; CA^2,$$

and $$QV^2 \; CP^2 - CV^2 = C\delta^2 . CP^2.$$

As a property of this "ellipse" we have

$$CP^2 + C\delta^2 = CA^2 + C\beta^2,$$

or $$CP^2 + (-CD^2) = CA^2 + (-CB^2),$$

which agrees with Art 52, except that strictly speaking we should say that the *sum* of the squares of any two conjugate diameters of a hyperbola is constant, the square of one of every two such diameters being negative and its length therefore imaginary In geometrical proofs every step has its explanation upon the figure. But let the student, shutting his eyes to the figure, regard any property of the ellipse (for example) as implicitly contained in the relation between its ordinates and abscissæ, and he will see that from any such property, in so far as it is expressible in terms of CB^2 and CD^2, he may pass at once to a corresponding

property of the hyperbola by writing $-CB^2$ for CB^2 and $-CD^2$ for CD^2 Thus, the radius of the orthocycle in an ellipse (Art 40, Cor 2) being equal to $\sqrt{(CA^2+CB^2)}$, its radius in the hyperbola is equal to $\sqrt{(CA^2-CB^2)}$ In the ellipse $SY\ S'Y'=CB^2$ (Art 40), therefore in the hyperbola $SY\ S'Y'=-CB^2$, or $SY\ (-S'Y')=CB^2$, one of the perpendiculars having to be regarded as positive and the other as negative because they are drawn in opposite directions, the tangent passing between the foci The same principles will be seen to be applicable to other properties for example, to the theorems of Arts 46 and 47

CHAPTER VI

THE EQUILATERAL HYPERBOLA

53 THE *Equilateral Hyperbola* is a hyperbola whose latus rectum is equal to its transverse axis Its two axes being equal (Art 34), its asymptotes are at right angles, and it is therefore called also the *Rectangular Hyperbola* Its eccentricity is the ratio of the diagonal to the side of a square, the foot X of the S-directrix bisects CS, and

$$\tfrac{1}{2}CS^2 = CA^2 = 2CX^2 = 2SX^2. \qquad \text{[Art. 32}$$

Also $$PN^2 = AN\ A'N = CN^2 - CA^2, \qquad \text{[Art 33}$$

and $$QV^2 = PV.P'V = CV^2 - CP^2 *, \qquad \text{[Art 36.}$$

since by Prop II every diameter is equal to its conjugate

The normal at P terminated by either axis is equal to CP or CD (Art 45, Cor 2), and therefore also to the intercept on the tangent between the curve and either asymptote [Art 50

It remains to prove certain of the more distinctive properties of this variety of the hyperbola, which bears the same kind of relation to the general hyperbola that the circle (or equilateral ellipse) bears to the general ellipse

PROPOSITION I.

54 *The angles between any two diameters of an equilateral hyperbola are equal to the angles between the conjugate diameters*

Let any two diameters meet the S-directrix in V and V'; and draw SZ at right angles to SV and SZ' at right angles

* This may also be deduced as a corollary from Prop I

to SV', so that SZ and SZ' are the directions of the diameters conjugate to CV and CV' [Art 13

Then since X is the middle point of CS, [Art 53

$$\angle VCV' = VSV' = ZSZ',$$

or the diameters CV and CV' are inclined at the same angles as their conjugates Thus *any two conjugate diameters* (Fig Art 55) make equal angles with the two axes they therefore *make equal angles with either asymptote, and make complementary angles with either axis*

Corollary 1.

Let PP' be any diameter, Q and R any two points on the curve, then since supplemental chords are parallel to conjugate diameters (Art 44) the angle QPR is equal or supplementary to $QP'R$, and therefore *any chord of an equilateral hyperbola subtends equal or supplementary angles at the extremities of any diameter.*

Corollary 2

In the second figure of Art 41 (supposing the hyperbola to be equilateral) it may be shewn by similar triangles that

$$QT \cdot Qt = CQ^2;$$

and likewise that $$CV.VT = QV^2,$$

or $$CV \cdot CT = CV^2 - QV^2 = CP^2$$

Corollary 3.

Let ABC be a given triangle, O the centre of an equilateral hyperbola circumscribed to it, and abc the middle points of its sides Then since Ob and Oc are the diameters bisecting the chords AC and AB respectively, they contain an angle equal to the given angle BAC The locus of O is therefore a circle passing through b and c, and likewise through a, since bc evidently subtends the given angle at a That is to say, *the locus of the centre of an equilateral hyperbola circumscribed to a given triangle is the nine-point circle of the triangle*

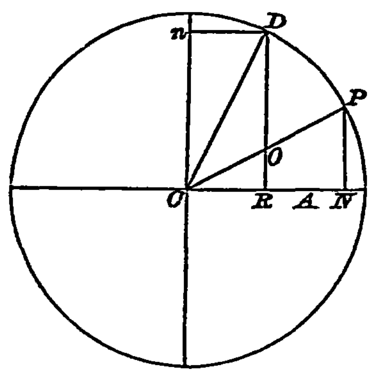
PROPOSITION II

55 *Conjugate diameters and diameters at right angles are equal in the equilateral hyperbola, and the ordinates and abscissæ of the extremities of any two conjugate semi-diameters are alternately equal to one another*

(i) The difference of the squares of any two conjugate diameters being equal to the difference of the squares of the axes (Art 45), and the axes being in this case equal, it follows that every diameter is equal to its conjugate

(ii) Let CP and CD be conjugate semi-diameters, PN and DR ordinates to the axis; and let PN produced meet the hyperbola again in P', so that $CP' = CP = CD$ Then

$\angle P'CD = PCN + DCN =$ a right angle, [Prop I



or any two equal semi-diameters CP' and CD lying in adjacent quadrants are at right angles, and conversely

(iii) Since the triangles CPN and CRD are similar (Prop I), and $CP = CD$, it readily follows that

$$PN = CR, \text{ and } DR = CN,$$

as was to be proved.

Corollary

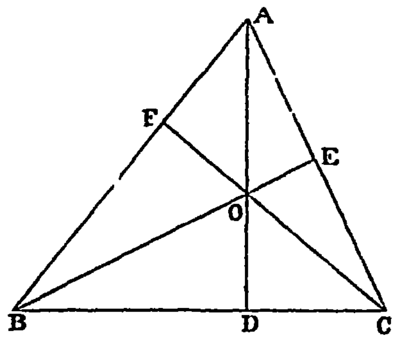
Hence and from the relation $CN^2 - PN^2 = CA^2$ it may be deduced that $\triangle CPD = \frac{1}{2}CA^2$ The conjugate parallelogram is therefore equal to $4CA^2$

PROPOSITION III

56 *If an equilateral hyperbola circumscribes a triangle it passes through its orthocentre, and conversely*

Let AD, BE, CF be the three perpendiculars of a triangle ABC, and O the point in which they cointersect, which is called the orthocentre of the triangle

Let a hyperbola be supposed to pass through the four points $ABCO$ Then since $AD\ DO = BD\ DC$, its diameters parallel to AD and BC are equal (Art 16, Cor 2), and in like manner its diameters parallel to BE and AC are equal, as also are those parallel to CF and AB



Hence the hyperbola through $ABCO$, having more than one pair of equal diameters at right angles, must be equilateral (Prop II), and it may be inferred conversely that every equilateral hyperbola circumscribing a triangle ABC must pass through its orthocentre O

The hyperbola was perhaps first discovered and traced from its asymptotes as the locus, $xy = $ a constant (*Anc and Mod G C*) It may be called *Acute*, *Obtuse* or *Right* according to the asymptote-angle which bounds it The *Rectangular* or *Rt* hyperbola (Fig Art 49) would have been discovered first and as such, and would afterwards have been seen to be "equilateral"

CHAPTER VII

THE CONE

57 AN unlimited straight line which passes through a fixed point in space and makes a constant angle with a fixed straight line through the point generates a surface which is called a *Cone* The fixed point is called the *Vertex*, the fixed line the *Axis*, and the variable line in any position is called a *Side* or a *Generating Line* of the cone The constant angle between any two opposite sides of the cone is called its *Vertical Angle* The complete cone consists of two infinite sheets situated on opposite sides of the vertex, as in the last figure of this chapter

This species of cone is more fully described as the *right circular* cone, the sections of it made by planes at right angles to its axis being evidently circles Any such section may be regarded as the *Base* of the cone.

In the special case in which the vertex is at infinity, and the generating lines are therefore all parallel to the axis and at right angles to the base, the surface is called no longer a cone but a *Cylinder*

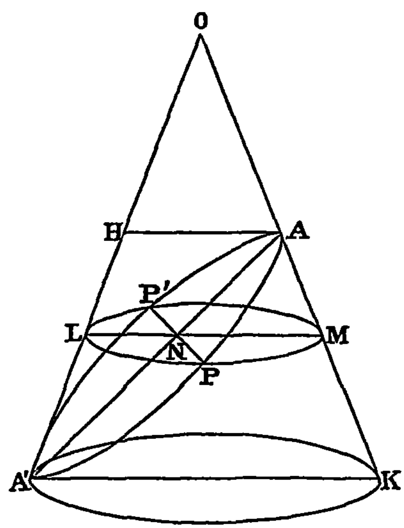
In what follows we shall consider a plane through the axis to be the *Plane of Reference* and the *Sections* to be made by planes at right angles thereto, so that the principal axis of a section will always lie in the plane of reference It will appear from the following propositions that a plane section of a cone is in general a parabola, an ellipse or a hyperbola

The Ordinate.

PROPOSITION I

58 *The square of the principal ordinate in any section varies as the rectangle contained by the corresponding abscissæ*

Let AA' be the axis of the section and PN the perpendicular upon it from any point P of the section Draw the



circular sections of the cone through P, A, A', and let their diameters in the plane of reference be LM, AH, $A'K$ respectively

Then by similar triangles

$$LN \quad A'N = AH \quad AA',$$

and

$$MN \quad AN = A'K \quad AA',$$

and by a property of the circle

$$PN^2 = LN \cdot MN$$

Therefore $\quad PN^2 \cdot AN \cdot A'N = AH \cdot A'K \cdot AA'^2,$

or the locus of P is an ellipse or a hyperbola, according as the plane of the section cuts all the generating lines of the cone on the same side of the vertex, as above, or cuts both sheets of the cone, as in the last figure of Art 59 In either case the conjugate axis is a mean proportional to AH and $A'K$, and therefore *the semi-axis conjugate is a mean proportional to the perpendiculars from the vertices of the section to the axis of the cone*

When the plane of the section is parallel to a side of the cone, it may be shewn in like manner that PN^2 varies as AN and the section is a parabola

Corollary

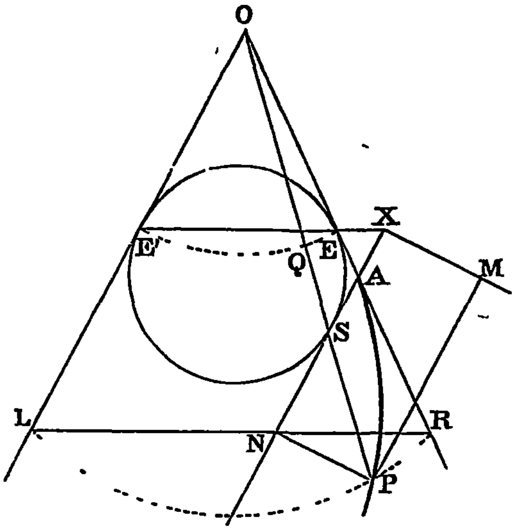
In the cylinder (in which all circular sections are equal) an oblique section is always an ellipse having its minor axis equal to the diameter of the base, whilst its major axis may be of any length greater than that diameter Conversely *any ellipse may be regarded as a plane section of a right cylinder described on a circular base equal to the minor auxiliary circle of the ellipse*

The Focal Spheres.

PROPOSITION II

59. *If a sphere be inscribed in a cone so as to touch the plane of a section, the point of contact of the sphere with the plane will be a focus of the section, and the plane of contact of the sphere with the cone will meet the plane of the section in the corresponding directrix*

(1) Let a sphere be drawn touching a cone along the circle EQE', and touching the plane of a section at the point S, then will S be a focus of the section, and the



corresponding directrix will be the line MX in which the plane of contact EQE' meets the plane of the section.

For let P be any point on the section, PY* a perpendicular to the plane of contact, PM a perpendicular to MX, and Q the point in which the side of the cone through P meets the plane of contact

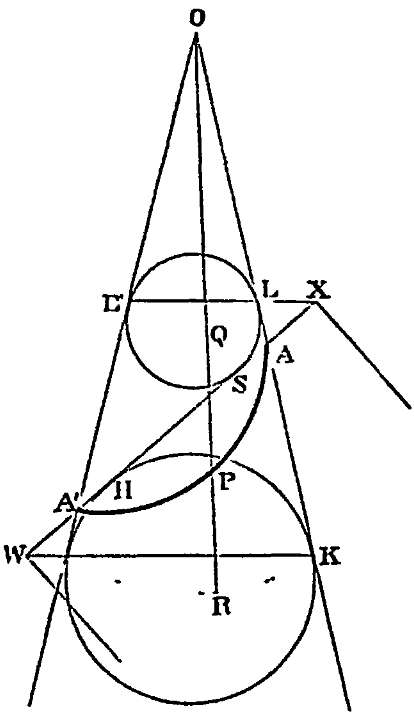
Then since the angle QPY is always equal to half the vertical angle of the cone, and the angle MPY to the angle between the axis of the section and that of the cone, it follows that PQ PY and $PY . PM$ and therefore also PQ PM are constant ratios

Hence, the tangents PS and PQ to the sphere being equal, SP also is in a constant ratio to PM, that is to say,

* This line is to be supplied in the diagram

the locus of P is a conic having the point S and the line MX for a focus and directrix

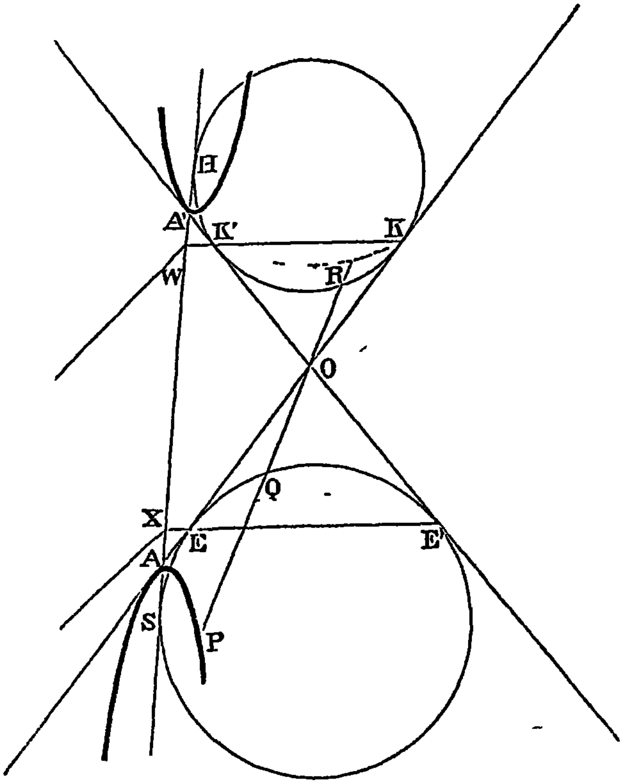
(ii) Every elliptic section has two focal spheres touching its plane on opposite sides, and every hyperbolic section two focal spheres touching its plane on the same side thus the two foci S, H and their directrices are determined in each case



Let the side of the cone through any point P of the section touch the focal spheres in Q and R, as in the annexed figures Then PS and PQ, being tangents to the S-sphere, are equal, and PH and PR, being tangents to the H-sphere are equal.

Hence, in the ellipse,

$$SP + HP = PQ + PR = QR$$



And in the hyperbola,

$$SP \sim HP = PQ \sim PR = QR$$

In either case QR is of constant length, and may be shewn, viz by making P coincide with A, to be equal to

$$AS \pm AH, \quad \text{or} \quad AA'$$

Corollary

Draw the tangent at P to the section and take any point T upon it Then PS and PQ, as being tangents to the S-sphere, are equal, and likewise TS and TQ are equal, and PT is common to the two triangles SPT and QPT Hence their angles at P are equal, or *at any point of a section the tangent makes equal angles with the focal distance and the side of the cone.*

CHAPTER VIII.

ORTHOGONAL PROJECTION

60. THE *Orthogonal Projection* or briefly the *Projection* of any point in space upon a plane, is the foot of the perpendicular let fall from the point to the plane. The projection of any line or figure is determined by the projections of its several points. It is evident that the projections of any figure upon parallel planes are equal and similar.

PROPOSITION I.

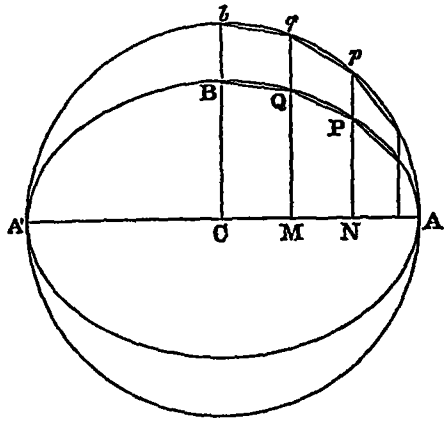
61. *Any ellipse may be projected into a circle equal to its minor auxiliary circle; and a circle may be projected into an ellipse of any eccentricity having its major axis equal to the diameter of the circle.*

(i) Describe a right cylinder upon a circular base equal to the minor auxiliary circle of the given ellipse; then the ellipse may be placed so as to coincide with one of the plane sections of the cylinder. [Art. 58, Cor.

The plane of the section is to be taken at such an inclination to the plane of the base that the projection of the major axis of the section upon the base may be equal to its minor axis.

(ii) All the ordinates to a diameter AA' in a circle are cut in the same ratio of minority $CB : CA$ by an ellipse whose major axis is AA' and whose minor axis is equal to $2CB$. [Art. 35.

Hence if the plane of the circle be turned about AA' through a certain angle*, every point P on the ellipse will



lie vertically under the corresponding point p of the circle, or the ellipse will be the orthogonal projection of the circle Thus the circle is projected into an ellipse having its major axis equal to AA', whilst its eccentricity increases with the acute angle between its plane and the plane of the circle and may be of any magnitude between zero and unity

PROPOSITION II

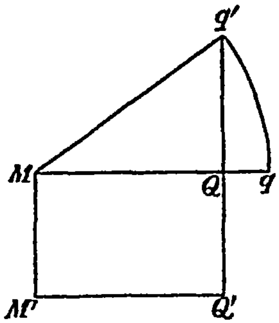
62 *The projections of parallel straight lines are parallel straight lines, and every line or segment in a system of parallels is in the same ratio to its projection.*

(1) The projection of a straight line upon a plane is the common section of that plane with the plane drawn at right angles to it through the line, since their common section evidently contains the projections of all points upon the line and of those only

Let Mq' be a finite straight line and $M'Q'$ its projection, so that the plane $Mq'Q'M'$ is at right angles to the plane of projection

* In both cases of the proposition the cosine of the angle between the two planes must be equal to CB CA

Then if Mq' be regarded as a variable line belonging to a system of *parallels*, the planes by which it is projected



will be parallel, and their common sections $M'Q'$ with the plane of projection will be parallel to one another, as was to be proved

(ii) Moreover, if MQ be drawn equal and parallel to $M'Q'$, the angle $q'MQ$ will be constant, and the ratio of Mq' to MQ or $M'Q'$ will therefore be invariable, as was to be proved

In the special case in which the original parallels and their projections are at right angles to the common section of their planes the constant angle $q'MQ$ is equal to *the angle between the planes*

PROPOSITION III.

63 *Any area in one plane is to its projection upon any other plane in a ratio which depends only upon the angle between the planes*

In the one plane draw any number of perpendiculars pN, qM,. to the common section of the planes, and let PN, QM, be their projections upon the other plane, so that

$$PN . pN = QM \;\; qM = CB \;\; CA,$$

where $CB \;\; CA$ is a ratio determined by *the angle between the planes** [Prop II

* Compare the figure of Art 61, supposing the planes of the two curves to be inclined at an angle whose cosine is equal to $CB \;\; CA$

It follows that every rectilinear figure $pNMq$ determined by the pair of perpendiculars pN, qM is to its projection $PNMQ$ as CA to CB, and the aggregate of any number of such figures is in the same ratio to the aggregate of their projections

But any rectilinear figure in the primitive plane may be divided into elements by means of perpendiculars drawn as above, and any curvilinear figure may be regarded as the limit of a rectilinear figure whose adjacent angular points are indefinitely near to one another Therefore any area whatever in the one plane is to its projection upon the other plane in the ratio of CA to CB, as was to be proved

PROPOSITION IV

64 *The points of concourse of lines and of their projections correspond to one another, and the tangent to a curve at any point corresponds to the tangent at the projection of the point*

(i) Since the projection of any line is determined by the projections of its several points, if any number of lines straight or curved pass through a point, their projections must all pass through the projection of the point For example, if a chord of any curve be drawn through a *fixed* point its projection will pass through the corresponding fixed point in the plane of projection

(ii) If a straight line and a curve intersect in adjacent points P and Q, the projections of the straight line and the curve will intersect at the projections p and q of those points Hence, the projecting lines Pp and Qq being always parallel, if the points P and Q coalesce their projections must also coalesce, that is to say, the tangent to the original curve at P projects into the tangent at p to the projection of the curve

We shall conclude by briefly indicating the method of applying these propositions, with reference in the first instance to the ellipse

65 *The Area of the Ellipse*

It may be shewn by projecting a circle into an ellipse that the area of any ellipse is to that of its auxiliary circle as CB to CA, and hence that the area of the ellipse is equal to $\pi\ CA\ CB$

66 *Conjugate Diameters*

The middle points of any system of parallel chords in a circle may be projected into the middle points of a system of parallel chords in an ellipse But in the circle parallel chords are bisected by a straight line, therefore in the ellipse also parallel chords have their middle points in a straight line Hence it appears that diameters at right angles in the circle correspond projectively to conjugate diameters in an ellipse

Hence a simple construction for drawing conjugate diameters to an ellipse

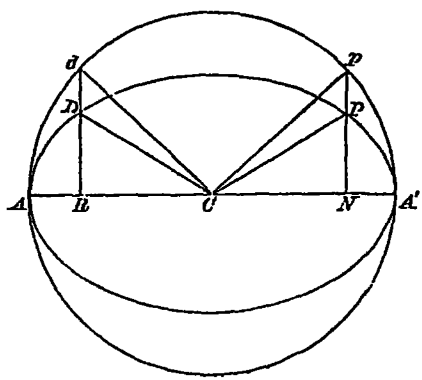
On the circumference of its auxiliary circle take points p and d which subtend a right angle at C, and let their ordinates to the axis, viz pN and dR, intersect the ellipse in P and D then will CP and CD be conjugate radii.

It is easily seen that pN is equal CR, and hence that

$$CN^2 + CR^2 = CN^2 + pN^2 = CA^2.$$

It then follows as in Art 45 (i) that

$$CP^2 + CD^2 = CA^2 + CB^2$$



67 *The Segments of Chords*

Project an ellipse into a circle, or a circle into an ellipse. Let POQ, $P'OQ'$ be any two intersecting chords of the ellipse and CD, CD' the parallel semi-diameters; poq, $p'oq'$ the corresponding chords of the circle and cd, cd' the parallel radii.

Then by Prop. II and by known properties of the circle,

$$\begin{aligned} OP \cdot OQ : CD^2 &= op \cdot oq : cd^2 \\ &= op' \cdot oq' : cd'^2 \\ &= OP' \cdot OQ' : CD'^2, \end{aligned}$$

or the rectangles contained by any two intersecting chords of an ellipse are as the squares of the parallel semi-diameters.

68 *The Tangent*

Any two tangents to a circle meet upon the diameter bisecting their chord of contact, viz. at a point T such that

$$CV \cdot CT = CP^2,$$

where V is the middle point of the chord, and P an extremity of the diameter CT. It readily follows by orthogonal projection that the same is true in the ellipse; and in like manner the property of Art. 47 may be first proved for the circle and then transferred by projection to the ellipse.

69 *Properties of Polars*

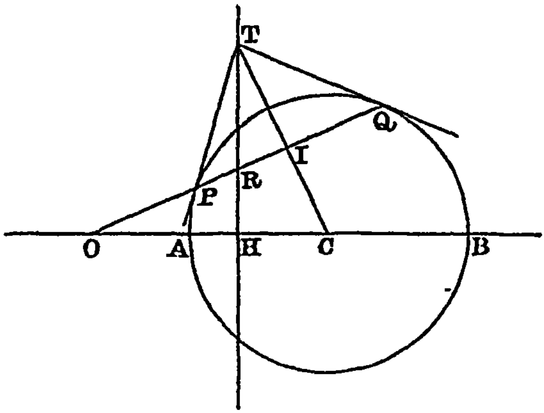
(a) *If a chord of a circle passes through a fixed point the tangents at its extremities intersect on a fixed straight line, and conversely.*

In a circle of radius CA draw any chord PQ through a point O, and let TP and TQ be the tangents at its extremities. Draw the perpendicular TH to CO, and let TH meet the chord in R, and let CT meet it in I.

Then since the angles at H and I are right angles the points H, I, T, O are concyclic, and

$$CH \cdot CO = CI \cdot CT = CA^2.$$

Hence if O be a fixed point, CH is constant and RH is a fixed straight line, and conversely



The line RH is called the *Polar* of O, and the point O is called the *Pole* of RH.

If the point O be taken without the circle its polar will be the chord of contact of the tangents from O if the point be taken within the circle its polar will lie wholly without the curve

(*b*) *Any chord of a circle is divided harmonically by any point through which it passes and the polar of the point*

For in the preceding figure the points C, I, R, H are concyclic, so that

$$\begin{aligned} OR\ OI = OH\ OC &= OC^2 \sim CO\ CH \\ &= CO^2 \sim CA^2 \\ &= OP.OQ \end{aligned}$$

Hence by Lemma B, since I is the middle point of PQ,

$$2OP\ OQ = 2OR\ OI = OR\,(OP + OQ),$$

and therefore PQ is divided harmonically at O and R

These properties of polars may be extended to the ellipse by orthogonal projection We may remark in passing that they are also true of the general conic. thus (to take a special case) the directrix is the polar of the focus [Art 7, Cor

70 *The Equilateral Hyperbola*

It readily follows from the property of the principal ordinate (Art 33) that an equilateral hyperbola may be projected into a hyperbola of any eccentricity, and *vice versa*

In the equilateral hyperbola let the length of any semi-diameter CD which does not meet the curve be *defined* by the condition that it shall be equal to the conjugate semi-diameter CP let it be granted further that

$$PN^2 \sim DR^2 = CN^2 \sim CR^2 = CA^2, \qquad \text{[Art 55}$$

and that the triangle CPD is equal to CA^2

It may then be deduced by projection that in the general hyperbola the difference of the squares of any two conjugate semi-diameters is equal to $CA^2 \sim CB^2$, and the area of the conjugate parallelograms to $AA' \cdot BB'$

In like manner the property of the ordinate to any diameter in the general hyperbola (Art 36) may be deduced from the special case of the equilateral hyperbola

CHAPTER IX

CURVATURE

71 It is evident that a circle and a conic cannot intersect in an odd number of points, and it may be inferred from Art 16, Cor 3, that they cannot intersect in more than four points

Let a circle and a conic intersect in four points $Q'PQP'$, two or more of which may be supposed to become coincident And first let Q' coalesce with P Then the circle and the conic have simple contact at P and intersect at Q and P', and their common tangent and their common chord QP' are equally inclined to the axis of the conic

Next let Q also coalesce with P Then the circle *both touches and cuts* the conic at P, and their common tangent and their common chord PP' are equally inclined to the axis of the conic

Lastly let P' also coalesce with P. Then the circle touches the conic without cutting it at P and does not meet it again

72 The circle which is the limit of a circle described so as to touch a conic at P and cut it at an adjacent point Q which ultimately coalesces with P is called the *Circle of Curvature* of the conic at P Its centre, radius and diameter are called the *Centre, Radius* and *Diameter of Curvature* at P, and its chord drawn from P in any direction is called the *Chord of Curvature* of the conic at P in that direction The

circle of curvature in general cuts the conic at their point of contact, but when this point lies on an axis of the conic the circle touches the conic without cutting it at that point and does not meet it again

PROPOSITION I

73 *The focal chord of curvature at any point of a conic is equal to the focal chord of the conic parallel to the tangent at that point*

Let a circle touch a conic at P and cut it at an adjacent point Q

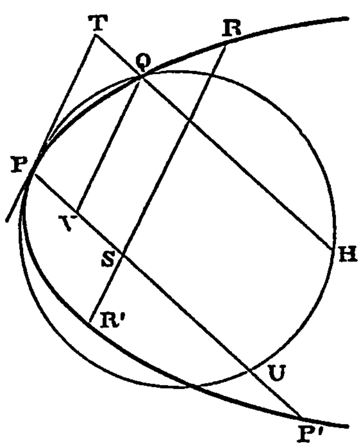
Draw the focal chord PSP' cutting the circle in U, and the focal chord RSR' parallel to the tangent at P, and let the parallel through Q to PP' meet the tangent in T, the circle in H and the conic in K

Then by Art 16, Cor 2,

$$TP^2 \quad TQ \quad TK = RR' \quad PP',$$

and by a property of the circle,

$$TP^2 = TQ \quad TH,$$



and therefore

$$TH \quad TK = RR' \quad PP'$$

Hence in the limit, when Q coalesces with P and therefore THK with PUP',

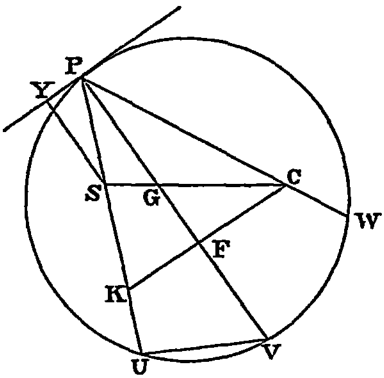
$$PU \cdot PP' = RR' \cdot PP',$$

or the focal chord of curvature PU is equal to the focal chord RR' parallel to the tangent at P

Corollary 1

If PV be the chord of curvature at P in any direction, and SY be drawn parallel to it to meet the tangent in Y, it may be shewn that

$$PV = \frac{SP}{SY} PU = \frac{SP}{SY} RR',$$



which determines the chord of curvature of a conic at any point in any direction

Corollary 2

It may now be deduced that in the Parabola, the chord of curvature at P through the focus, or parallel to the axis, being equal to $4SP$ (Art 21), the diameter of curvature is equal to

$$\frac{4SP^2}{SY}, \text{ or } \frac{4SY^3}{SA^2},$$

where SY is the focal perpendicular upon the tangent at P, and in the Central Conics that, the focal chords of curvature being equal to $\frac{2CD^2}{CA}$ (Art. 36, Cor), the central chord of curvature is equal to $\frac{2CD^2}{CP}$ (Art 39, Cor 2), and the diameter of curvature to $\frac{2CD^3}{CA\ CB}$, or $\frac{2CD^2}{PF}$ (Art 46)

Corollary 3

The focal chord of curvature at P in any conic is equal to $\frac{S'P}{2CA}$ $4SP$ (Art 45, Cor 1), and the ratio $\frac{S'P}{2CA}$ is equal to or less or greater than unity according as the conic is a parabola, an ellipse or a hyperbola Conversely *a conic is a parabola, an ellipse or a hyperbola according as its focal chord of curvature at any point P is equal to or less or greater than* $4SP$*, the point P being supposed to lie on the S-branch in the case of the hyperbola This may also be proved by shewing that SG is equal to or less or greater than SP according as S, in Art 75, lies at the middle point of PU, or at a greater or less distance from P

PROPOSITION II

74 *At any point of a conic the chord of curvature in any direction is to the chord of the conic in the same direction as the focal chord parallel to the tangent is to the focal chord parallel to the chord of curvature*

Let a circle meet a conic in three adjacent points QPQ', and let a chord PU of the circle meet QQ' in V and the conic

* The square of the velocity V at any point P of an orbit described under the action of a central force F being measured by

$2F \times \frac{1}{4}$ (chord of curvature through centre of force),

a conic so described about S will be a parabola, an ellipse or a hyperbola according as V^2 is equal to or less or greater than $2F\ SP$ See Newton's *Principia*, Lib I prop 6 (sect 2), and prop 17 (sect 3)

again in P' Then it may be shewn by the method of Prop I that VU is to VP' as the focal chord parallel to PP' is to the focal chord parallel to QQ' The required result follows by making the points Q and Q' coalesce with P, in which case the circle evidently becomes the circle of curvature as defined in Art 72

Thus the chord of curvature in any direction is determined For example the central chord of curvature is at once seen to be equal to

$$\frac{CD^2}{CP^2} \cdot 2CP, \quad \text{or} \quad \frac{2CD^2}{CP}$$

75 *A construction to determine the centre of curvature at any point of a conic*

From Art 73, Cor 2, it readily follows that at any point of a conic,

$$\text{radius of curvature} = \frac{(\text{normal})^3}{(\frac{1}{2}\text{ lat rect })^2}$$

Hence the following construction for the centre of curvature at P

In the figure of Art 12 draw GU at right angles to PG to meet SP, and UO at right angles to SP to meet PG Then

$$\frac{PO}{PU}\,\frac{PU}{PG} = \frac{PG^2}{PK^2},$$

or

$$PO = \frac{PG^3}{PK^2},$$

and therefore PO is the radius and O the centre of curvature at P

In this construction PU is the semi-chord of curvature through S, and is equal to $\frac{PG^2}{PK}$

CHAPTER X

THE LINE INFINITY

76 *The direction of the line Infinity is indeterminate*

For if a straight line drawn in any given direction in a plane be removed to infinity, it will evidently coalesce with the line Infinity (Def 12), and will contain all points upon it since opposite points at infinity coincide

[Lemma H and Prob 395

The line Infinity may therefore be regarded at every point upon it as parallel or inclined at any angle to any straight line*

Thus in the Parabola, to which it is a tangent (Art 26 or Prob 395), it may be regarded as at right angles to every other tangent (p 44)

The line Infinity, while indeterminate in direction, is to be regarded in other respects as a single determinate straight line Thus it meets every curve of the nth degree in n single or an equivalent number of multiple points, and conversely a curve which has such number of points at infinity is of the nth degree

77. *Every Line-Circle contains the line Infinity*

For when the centre of a circle drawn through two fixed points is removed to infinity, the circumference evidently contains all points at infinity in its plane as well as the unlimited line joining the fixed points

* Otherwise thus By an extension of Euclid xi 3, the Infinity locus which is the section of a plane by a parallel plane is to be regarded as a straight line, and this line may evidently be regarded as the limiting position of a line drawn in any given direction

The Focoids.

78 *All circles in a plane pass through the same two imaginary points on the line Infinity*

Let ϕ and ϕ' be the two imaginary points in which a given circle meets the line Infinity* These will be the same for all circles in the plane

For an arc AB of the circle which subtends any angle α at the circumference subtends that angle at ϕ and ϕ' But, these points being on the line Infinity, any two lines ϕA and $\phi A'$ through either point may be regarded as parallel, and likewise any two lines ϕB and $\phi B'$, so that

$$\angle A'\phi B' = A\phi B = \alpha$$

That is to say, the points ϕ and ϕ' are such that *any two straight lines through either may be regarded as intersecting at any angle*

Considering any other circle, take on it points A and B The circle is the locus of the points at which A and B subtend a certain angle α. But at ϕ they subtend an angle α Therefore ϕ is a point of the circle And similarly ϕ' is a point of the circle

The points ϕ and ϕ' have been called the *Circular Points at Infinity* We shall call them the *Focoids,* on account of their relation to the foci of conics and other curves

Every conic through the Focoids is a circle For, the circle determined by any three other points on such conic passing also through the focoids, the conic meets it in five points, and therefore coincides with it

[Art 71 and Prob 400

79 *A straight line through a Focoid may be regarded as inclined at any angle to itself*

For any two such lines, which contain an indeterminate angle (Art. 78), may be supposed to coalesce

* The letter ϕ, while as the equivalent of F it stands for *Focoid,* serves also as a symbol of the intersection of a circle by the line Infinity

Conversely, any line which makes a finite angle with itself must pass through a Focoid

80 *The factors of a Point-Circle*

A circle being the locus of the vertex of a given finite angle whose arms pass each through a fixed point, when the fixed points coalesce the locus becomes a pair of straight lines through the Focoids. [Art 79

That is to say, the Point-Circle at any point O is made up of the lines $O\phi$ and $O\phi'$

Every point P on either line satisfies the relation,

$$OP=0$$

since when the given angle at P is a right angle, OP is always equal to half the distance between the fixed points

As an exercise in the properties of the Focoids see the article on *Orthoptic Loci* at the beginning of Vol XVI of the *Messenger of Mathematics*

The Foci.

81 *The tangents to a conic from its Foci are the lines joining them to the Focoids*

We assume that a directrix of a conic is the chord of contact of the imaginary tangents to it from the corresponding focus [Art 69

If now the point of contact P in Art. 9 be one of the two imaginary points in which the directrix cuts the conic, so that PR vanishes while still subtending a right angle at S, it follows that SP, which by hypothesis is the tangent to the conic at P, must pass through a Focoid [Art 79

Hence the conic has double contact with the point-circle at either focus, this being made up of the tangents to the conic from that focus [Art 80

The foci of higher curves are defined as the opposite intersections of the tangents to them from ϕ and ϕ'

If the foci of an ellipse be so defined, it will have in addition to its foci S and S', two imaginary foci Σ and Σ'. Their positions may be deduced from those of S and S' by interchanging the axes: that is to say, by taking points Σ and Σ' on the minor axis, such that

$$C\Sigma = C\Sigma' = \sqrt{(CB^2 - CA^2)}$$

To pass to the case of the hyperbola we have only to write $-CB^2$ for CB^2. [Scholium, p 70

All conics which have the same foci S and S' touch the same four lines $S\phi$, $S\phi'$, $S'\phi$, $S'\phi'$, and all concentric circles touch one another at ϕ and ϕ'

Every curve which touches the line Infinity has a pair of its Foci at the Focoids

Scholium.

Any circle in a plane being represented in rectangular coordinates by the general equation,

$$x^2 + y^2 + Ax + By + C = 0,$$

all such circles pass through the same two imaginary points ϕ and ϕ' at infinity as the point-circle whose equation is

$$x^2 + y^2 \equiv (x + iy)(x - iy) = 0 \qquad \text{[Art 80}$$

But when we attempt to determine ϕ and ϕ' by means of the equation $x^2 + y^2 = 0$, we are met by the difficulty that the radius vector to either makes a certain angle θ, such that $\tan\theta = \pm i$, with the axis of x *whatever be its direction* It seems to follow that the direction of the radius vector to ϕ or ϕ' and the positions of those points at infinity are indeterminate. The reason why this does not follow is that the angle θ is so far indeterminate that

$$\tan(\theta + \alpha) = \frac{\pm i + \tan\alpha}{1 \mp i\tan\alpha} = \pm i = \tan\theta$$

If therefore a line be such that it makes an angle $\tan^{-1} \pm i$ with any axis, it makes the like angle with any other axis This makes it possible for ϕ and ϕ' to be points which have each one determinate position, and they must be such points if every circle (not being a line-circle) is to have two points only at infinity

Notice in Art 81 that the relation,

$$C\Sigma=\sqrt{(CB^2-CA^2)}=\pm i\ \ CS,$$

is consistent with the property of every line $S\Sigma$ through a Focoid, that it makes an angle $\tan^{-1}\pm i$ with any other line

Since $$\tan\theta\equiv\frac{1}{i}\ \frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}=\pm i,$$

therefore $e^{i\theta}=0$ or ∞, or $i\theta=\mp\infty$, and therefore $\theta+a$ or θ is of the form $a\pm i\ \ \infty$ In other words the anomalous equation,

$$\tan(\theta+a)=\tan\theta=\pm i,$$

signifies that the tangent of the complex angle $a\pm i\ \ \beta$ approaches asymptotically to the limit $\pm i$ as β becomes infinite

A straight line through a Focoid is commonly said to be only *at right angles* to itself, because in the expression for the tangent of the angle between $y=mx$, and $y=m'x$, the denominator $1+mm'$ vanishes when $m=m'=\pm i$ But the numerator $m-m'$ also vanishes, and the angle is indeterminate [Art 79

CHAPTER XI

A NEW TREATMENT OF THE HYPERBOLA

82 *The asymptotes* Use Fig Art 49, adding Y, Y'

Having proved that

$$CS \quad CA = CA \quad CX = \text{the eccentricity}, \qquad [\text{Art } 32$$

with centre C and radius CA describe a circle cutting the directrix in Y and Y' Then

$$CS \,.\, CX = CA^2 = CY^2,$$

and the angle CYS is a *right angle*, and SY touches the circle at Y The lines CY, CY' are called the asymptotes*

83 *Construction for points on the curve*

Produce PN, the ordinate of any point P on the curve, to meet the asymptotes in p, p'. Then (fig Art 85)

$$pY \quad NX = CY \quad CX = \text{the eccentricity} \qquad [\text{Art } 82$$

$$= SP \quad NX, \qquad [\text{Def } 1$$

or *the intercept* pY *is equal to the focal distance* SP Hence a circle round S of radius equal to pY cuts pp' in points P, P' above and below the axis lying on the hyperbola Real points are thus determined when the ordinate is

* The construction of Art 6 gives CY, CY' as the tangents from C to the hyperbola Their points of contact are at infinity (Art 7), the focal chords at right angles to SY, SY' being parallel to CY, CY'

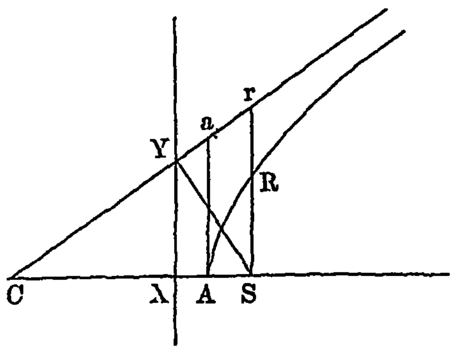
drawn to the right of A or to the left of the further vertex or intersection of the curve with the axis*.

84 *The latus rectum and the axes*

The tangent at the vertex A meets the asymptotes in points a, a' such that, CY being equal to CA,

$$Aa = Aa' = SY,$$

and $$Ca = Ca' = CS$$



Draw the Semi-latus Rectum SR, and produce it to meet CY in r Then, CYS being a right angle (Art 82),

$$SY^2 = rY\ CY = rY\ CA = SR \, . \, CA. \qquad \text{[Art 83}$$

Define the conjugate axis BB' as equal to the tangent aa', and therefore of the length $2SY$ Then *CB is a mean proportional to SR and CA* The triangle CYS gives the relations,

$$CS\ SX = CB^2, \quad CS^2 = CA^2 + CB^2$$

85 *The principal ordinate and abscissa*

Produce PN to meet CY in p, as in Art 83

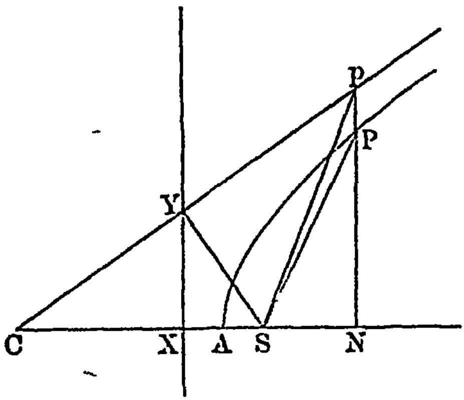
Then $\quad SY^2 + pY^2 = Sp^2 = pN^2 + SN^2 \qquad$ [Art 82

$$= pN^2 + SP^2 - PN^2,$$

* It will be seen that as pY increases from zero to infinity SN (which for real points must be not greater than pY) either increases from SX, but less rapidly than pY, or first decreases to zero and then increases.

or, cancelling equals pY^2 and SP^2 (Art 83),

$$SY^2 = pN^2 - PN^2.$$



Hence $\quad PN^2 + SY^2 = pN^2 = \dfrac{SY^2}{CY^2}\ CN^2,$

or $\quad PN^2 + CB^2\ \ CN^2 = CB^2\ \ CA^2$ [Art 84

86. *The focal distances* Use fig Art 49, adding Y, Z

Art 85 shews the symmetry of the curve and the consequent existence of a second focus S'* Let the asymptote CE meet the directrices as in Y and Z. Then by Art 83,

$$S'P - SP = EZ - EY = YZ$$
$$= 2CA$$

The focal distances $S'P$, SP are equal to $CE \pm CA$ and contain a rectangle equal to $CE^2 - CA^2$

Hence $\quad SP\ S'P + CA^2 = CE^2 = EN^2 + CN^2$

$$= CP^2 + CB^2,$$ [Art 85

which reduces to $SP.S'P = CD^2$, by Art 52

* Or thus, from $PN^2 + SY^2 = EN^2$, without using the principle of symmetry (Art 11) Find S' beyond the centre by drawing a parallel to CY at distance SY, and draw a perpendicular $S'Z$ to CY Then shew that, $S'Z$ being equal to SY, $EZ^2 = S'E^2 - SY^2 = S'E^2 - EN^2 + PN^2 = S'P^2$

87 *The curve in relation to its asymptotes*

Since $CB^2 = EN^2 - PN^2 = EP(EN+PN)$, [Art 85

the curve as it spreads out continually approximates to its asymptotes, EN and PN both increasing and EP therefore decreasing indefinitely Art 85 gives also,

$$PE\ PE' = EN^2 - PN^2 = CB^2.$$

88 *Of chords in general*

In the figure of Art 50 draw the ordinate of Q or q, and let it meet the asymptotes in E and E' Then, $QE\ QE'$ or $qE\ qE'$ being equal to CB^2, the rectangle $QR\ Qr$ or $qR\ qr$ is of a certain constant magnitude so long as Qq remains parallel to itself (Art 49, III) Call this constant CD^2, and bisect Rr in V. Then it is easily seen that

$$RV^2 - QV^2 = CD^2 = rV^2 - qV^2, \quad \text{[Euc II 5, Cor}$$

and hence that $RQ = rq$, and that, if Qq be parallel to the tangent at P, then $PT = Pt = CD$, and that parallel chords as Qq are bisected by a diameter CP, and that

$$\begin{aligned} QV^2 + CD^2\ \ CV^2 &= RV^2\ \ CV^2 \\ &= CD^2\ \ CP^2 \end{aligned}$$

The results of Art 16 may be deduced by the method of Art 97

89 *Conjugate diameters*

In the same figure, let the diameter DD' parallel to the tangent at P be defined as equal to Tt, and let PN and DR be the ordinates of P and D (as in Art 45), and let them meet the asymptote CT in p and d Then, PD being parallel to one asymptote and bisected by the other, it may be shewn that Pd and pD are parallel to the axis, so that $PN = dR$, and $DR = pN$, or

$$PN\ \ CR = DR\ \ CN = CB\ \ CA$$

Hence and by Art 85,

$$PN^2 \sim DR^2 = CB^2, \quad CR^2 \sim CN^2 = CA^2$$

By subtraction, $CP^2 \sim CD^2 = CA^2 \sim CB^2$* [Euc I. 47

90 *A chord parallel to an asymptote meets the hyperbola in one point only at a finite distance*

If a chord PQ meet the directrix in M,

$$SQ : QM = SP : PM$$ [Art 5

If PQ be parallel to an asymptote and $SP = PM$ (Art 83), then $SQ : QM$ is a ratio of equality This is the case when Q is *at infinity*, and it is not otherwise possible For, if SQ were finite and equal to QM, then $SP \pm SQ$ would be equal to PQ, or two sides of a triangle together equal to the third side

Scholium.

If lines drawn in opposite directions be regarded as positive and negative respectively, and if imaginary points (Scholium, p 70) be treated as subject to the same laws as points which can be marked on the figure, reason may be shewn why one of every two conjugate diameters of a hyperbola should have its square negative In Art 51, with the figure as drawn, the rectangle $RQ \cdot Rq$ is equal to a given magnitude CD^2 for all chords Qq drawn in the same direction, and when CV is less than CP the points Q and q become imaginary Call them δ and δ' when CV vanishes In this case R coincides with C and the rectangle becomes $C\delta \cdot C\delta'$, which must accordingly be equal to CD^2 But the semi-diameters $C\delta$ and $C\delta'$, being opposite in direction, are of opposite signs, whereas RQ and Rq in the figure are of the same direction and sign Therefore $C\delta^2 = -C\delta \cdot C\delta' = -CD^2$, or the square of the diameter parallel to Qq is negative

* A simpler proof is given in Art 52, for which the conjugate hyperbola (Art 99) shewn in the figure is not wanted In drawing a fresh figure make LCM by preference an acute angle

CHAPTER XII

A COURSE FOR BEGINNERS

[Some students in their first reading of the subject may find it best (after reading Art 6) to take the three conics separately, as in this chapter It will serve as an exercise to complete the duplicate proofs here given in outline]

THE PARABOLA

Read Chap III with the parts of Chap II referred to in it, or with Art 91 instead of Arts 7—9 The Corollaries of Art 13 may be read as belonging to Art 20

91 *The tangent according to Euclid's definition*

In Fig Art 24 draw the line bisecting the angle SPM, and take any point t upon it other than P Then St (or tM) is greater than the perpendicular distance of t from the directrix, and (Prob 1) t lies without the curve Thus the bisector of the angle SPM meets but does not cut the parabola at P, and is therefore the tangent at P, and conversely

It follows that the angle PSR is a right angle

Any point on the tangent at P is equidistant from SR and the directrix Hence (Art 6, end) the two tangents from a given point can be drawn and can be shewn to subtend equal angles at S

92 $QV^2 = 4SP\ PV$ (*Art* 22) *deduced from Art* 30

By Art 30 and similar triangles SPR, RPT (Art 28),

$$QV^2 = 4PR^2 = 4SP\ PT = 4SP\ PV$$

But the proof by chord-properties is to be preferred, the more so because of the converse in Art 22, Cor, which is wanted in Elementary Mechanics

93 *Equal focal chords are equally inclined to the axis*

For in Art 21, when the length SP is given there are two positions of SP equally inclined to the axis, and two such of SO and of QQ' at right angles to SO Art 16, Cor 3, so far as it applies to the parabola, may be read as a Corollary to Art 23

THE ELLIPSE.

Read Chaps II. and IV. as chapters on the Ellipse, omitting all references to the Hyperbola Or begin with Art *47 (p. 64) In Art 8, the ratio of ST to TN being greater than the eccentricity, the line PR may be shewn to be the tangent at P according to Euclid's definition

94 *The latus rectum and the axes*

Draw the minor semi-axis CB, and draw BM, SY perpendicular to the directrix and BM Shew from Def 1 and Art 32 that $SB = CA$, and that the angle BSM is a right angle The triangle BSM shews the relations between the segments of the axis, the minor axis and the latus rectum

Thus $$CS\ SX = SY^2 = CB^2,$$

and (Def 1), if SR be the Semi-latus Rectum,

$$SR\ CA = CS\ SX = CB^2$$

Compare Art. 84, noticing the coincidence that an asymptote (like BM) is a tangent equidistant from the two foci

95. *The ordinate* (*Art* 33) *Proof from Art* 13

Draw PA, PA' (Art 33), and let the diameter of AP meet the directrix in V Then shew that

$$PN\ AN = SX\ VX,$$

and $$PN\ A'N = VX\ CX$$

Hence $PN^2\ AN\ A'N = CS\ SX\ CS\ CX = CB^2\ CA^2$

96 $QV^2\ CP^2 - CV^2 = CD^2\ CP^2$ *Proof from Art* 41

In Fig. Art 43, which suits the case of the axis,

$$PN\ Ct = NT\ CT = CT - CN\ CT,$$

or PN^2 is to $PN\ Ct$ (or CB^2) as $CN\ CT - CN^2$ to $CN\ CT$,

or $$PN^2 . CB^2 = CA^2 - CN^2\ CA^2.$$

The letters only have to be changed for the general case as stated in the enunciation This may also be written in the form,

$$CV^2\ CD^2 - CV^2 = CP^2\ CD^2$$

97 *Of chords in general* (*Art* 16) *Proof from Art.* 96

In Fig Art 36 let QR be any chord through a point O and V its middle point, CD the parallel radius and CP its conjugate

Draw CO meeting the curve in q and draw qv an ordinate to CP Then because QV, qv are ordinates to CP,

$$CD^2 - QV^2 \quad CD^2 - qv^2 = CV^2 \quad Cv^2 = OV^2 \quad qv^2,$$

which is equal to CO^2 Cq^2, and is therefore a constant ratio so long as O is a fixed point

But $CD^2 - QV^2 + OV^2$ is to $CD^2 - qv^2 + qv^2$ in the same ratio OV^2 qv^2

That is to say, $CD^2 - QO$ OR varies as CD^2, and hence QO OR varies as CD^2, and therefore, if $Q'R'$ be any other chord through O and CD' the parallel radius,

$$QO \quad OR \quad CD^2 = Q'O \quad OR' \quad CD'^2$$

98 *Any two tangents* TP, TQ *make equal angles with the focal distances* ST, HT *respectively* (*Art* 40, *Cor* 1)

a For a circle round T touches SP, SQ, HP, HQ (*Anc and Mod G C Art* 49), and, given two tangents SP, HQ to a circle, their intercept on any third subtends a constant angle at the centre Hence SQ,HP subtend equal angles at T

b Or if SQ, HP meet in O, and HQ be produced to a point R, as in the same figure, then by Euclid I 32 and Arts 9 and 38,

$$\angle HTQ = \tfrac{1}{2}(OQR - OHQ) = \tfrac{1}{2}HOQ = STP^*$$

The Hyperbola

Read Chap XI for chord-properties, and Chapters II and IV chiefly for tangent-properties, drawing special figures for the Hyperbola To draw the normal at P, make CpG (Arts 85, 43) a right angle, and join PG

99 *The two branches of a hyperbola make up one curve*

This is shewn in the note on Art 49, Cor. (p 67)

The diameters of a hyperbola which do not meet it in real points are sometimes said to be terminated by the conjugate hyperbola (Prob 259) But this brings in an unnecessary curve and is apt to mislead the student On the true lengths of such diameters see the Scholium to Chap XI

* In the hyperbola the angles STP and HTQ are supplementary or equal according as the tangents are drawn to the same branch or opposite branches But in all cases *the perpendiculars from the foci to the tangents, each to each, subtend equal angles at their point of concourse*

PROBLEMS.

The General Conic.

1 The distance of any point outside a conic from the focus is to its perpendicular distance from the directrix in a ratio greater than the eccentricity, and conversely

2 If an ellipse, a parabola, and a hyperbola have the same focus and directrix, the ellipse lies wholly within the parabola, and the parabola wholly within the hyperbola

3 Conics having the same focus and directrix do not meet

4 If SL be the semi latus rectum, and SD be drawn parallel to PR (Art 4) to meet the directrix, then

$$SP \quad PR = SL \quad SD$$

5 The segments of any focal chord subtend equal (or supplementary) angles at the foot of the directrix

6 Determine the pole of the latus rectum and the polar of the focus [Art 69

7 If chords PR, QR be produced to meet the directrix in p, q, the angle between the focal radii to p, q will be equal (or supplementary) to half the angle between the focal radii to P, Q

8 If two tangents TP, TQ meet any third tangent in p, q, the angle between the focal radii to p, q will be equal (or supplementary) to half the angle between the focal radii to P, Q

9 The lines joining the extremities of any two focal chords of a conic intersect on the directrix, and the focal distances of their intersections are at right angles

10 If the directrix be removed to an infinite distance from the focus, as in Def 1, the conic becomes a circle Find what properties of the circle are contained in the theorem of Art 8 and in Probs 7, 8, 9

11 If the focus and two points of a conic be given, the directrix must pass through one of two fixed points

12 Having given two points on a conic and its focus and eccentricity, shew how to describe the curve

13 Having given two points on a conic and its directrix and eccentricity, shew how to describe the curve

14 Having given the focus and three points of a conic, shew how to describe the curve

15 If PN be the principal ordinate of any point P of a conic, then

$$SP \pm SL \quad SN = \text{the eccentricity},$$

where SL is the semi-latus rectum

16 Determine the condition that a chord of a conic may be greater than, equal to, or less than the diameter of the eccentric circle of its middle point

17 If the point p (Art 6) describe a series of circles about the same centre O, the point P will describe a series of conics having the same focus and directrix, and the eccentricities of the conics will be to one another as the radii of the circles

18 If pm be drawn perpendicular to the directrix (Art. 6), then $pm \quad PM = OD \quad SX$ Hence shew that every focal chord is divided harmonically by the focus and directrix

19 At any point of a conic the tangent makes a greater or less angle with the focal distance than with the perpendicular to the directrix according as the eccentricity is a ratio of minority or of majority

20 If two conics have a common focus, their common chord or chords will pass through the point of concourse of their directrices

21 If they also touch each other and the tangent at the point of contact be drawn, and if from any point on this common tangent second tangents be drawn to the conics, the line joining their points of contact will pass through the focus

22 If the tangent at any point of a conic meet the directrix in D, and the latus rectum in L, then

$$SL \quad SD = SA \quad AX$$

23 The tangents at the ends of a focal chord meet the latus rectum at points equidistant from the focus

24 The focal distance of any point on a conic is equal to the ordinate of the point produced to meet the tangent at an extremity of the latus rectum

25 If a chord of a conic subtend a constant angle at the focus, its envelope and the locus of its pole are conics having the same focus and directrix, and the eccentricities of the three are proportionals [Art 8

26 The vertex of a circumscribed triangle whose base subtends a constant angle at the focus lies on a conic having the same focus and directrix. [Prob 8

27 If TP be a tangent from T to a conic, and if the ordinate of T meet the curve in Q, the projection of ST upon SP is equal to SQ

28 Shew that PN and QM in Art 15 meet on the axis

29 Prove that the line PR in Art 8 is a tangent according to Euclid's definition Prove also that Prop II may be extended to chords as follows —If O be any point on a chord PQ, whose pole is T, and M be the projection of O upon the directrix, and if the perpendicular from O to ST meet SP, or SQ, in L, then

$$SL \quad OM = SA \quad AX$$

30 Given the focus of a conic and a focal chord, the locus of the extremities of the latus rectum is a circle

31 Given a chord of a conic and the angle which it subtends at the focus, the focal distance of its pole passes through a fixed point

32 Any two tangents to a conic intercept upon a tangent drawn parallel to their chord of contact a length which is bisected at the curve

33 The portion of any tangent to a conic intercepted between the tangents at the ends of the parallel focal chord subtends a right angle at the focus, and is divided by its point of contact into segments equal to the distance of that point from the focus

34 If M be the projection upon the directrix of any point P on a conic, shew that SM meets the tangent at the vertex upon the bisector of the angle SPM

35 Two sides of a triangle being given in position, if the third subtend a constant angle at a fixed point determine its envelope [Prob 26

36 If a fixed straight line intersect a series of conics which have the same focus and directrix, the envelope of the tangents to the conics at the points of section will be a conic, having the same focus, and touching both the fixed straight line and the directrix of the series of conics

37 If SY be the focal perpendicular on the tangent at P, and X the foot of the directrix, then

$$SY \quad YX = SA \quad AX$$

Determine the locus of Y, and shew that it is the envelope of the circle on SP as diameter

38 Shew also that XY meets the latus rectum at the foot of the perpendicular to it from P

39 If the diameter at a point P on a conic bisect the chord normal at Q, the diameter at Q bisects the chord normal at P

40 The normal PG (Art 11) becomes equal to the semi-latus rectum when P coalesces with the vertex

41 The perpendicular from G to SP varies as the ordinate of P, and the line joining the foot of this perpendicular to the foot of the ordinate of P is parallel to SM

42 If Q be any point on the normal at P, and if QL be a perpendicular to SP, and QM a perpendicular to the ordinate of P, then

$$QL \quad PM = SA \quad AX$$

43 The perpendicular to any focal chord from the intersection of the normals at its extremities meets the chord at a distance from one extremity equal to the focal distance of the other, and the parallel to the axis from the intersection of the normals bisects the chord

44 If P be the pole of any normal chord of a conic, and Q the point in which it meets the directrix, the circle SPQ passes through an extremity of the chord

45 A circle which touches a conic and has its centre upon the axis intercepts a constant length upon the focal radius to either point of contact

46 If QQ' be the focal chord at right angles to the normal PG, then

$$PG^2 = SQ \; SQ' \qquad \text{[Prob 33, Art 15}$$

47 Construct a conic of which an arc is given

48 The parallel diameters of two similarly situated central conics of the same eccentricity bisect the same systems of parallel chords If the two conics be concentric ellipses or hyperbolas (or equal parabolas whose axes are coincident), shew that any

chord of the exterior conic is divided into pairs of equal segments by the interior, and that any chord of the former which touches the latter is bisected at the point of contact

49 The angle between any two chords of a conic is equal to the angle subtended at the focus by the portion of the directrix intercepted by the diameters which bisect the chords

50 The polar of any point with respect to a conic meets the directrix on the diameter which bisects the focal chord through that point

51 Any chord of a conic and the diameter which bisects it meet the axis and the directrix respectively upon a line parallel to the focal distance of the pole of the chord Hence shew that in a central conic the polar of any point meets the axis at a distance from the centre which varies inversely as the central abscissa of the point.

52 The triangle whose vertices are the focus of a conic and the intersections of the tangent and the diameter at any point with the axis and the directrix respectively has its orthocentre at the point in which the tangent meets the directrix

53 Given the focus and directrix of a conic, shew that the polar of a given point with respect to it passes through a fixed point

54. Deduce from Art 16 that the square of the ordinate at any point of a conic varies either as the distance of the foot of the ordinate from the vertex, or as the rectangle contained by the segments into which it divides the axis

55 Any focal chord of a conic and the diameter which bisects it meet the directrix (or any fixed straight line perpendicular to the axis) at points whose ordinates contain a constant rectangle Hence find the locus of the middle point of a focal chord Determine also the locus of the foot of the perpendicular in Prob 43

56 Find the locus of the middle point of a chord of a conic which passes through a fixed point in the axis

57. Deduce from Art 10 that a line drawn from any point in the directrix of a conic is cut harmonically by the point, the curve, and the polar of the point

58. If OTO' touch a conic in T, and if OPQ, $O'P'Q'$ be a pair of parallel chords, then

$$OT^2 : O'T^2 = OP \cdot OQ : O'P' \cdot O'Q'.$$

59 Hence shew that a line drawn through *any* point is divided harmonically by the point, the curve, and the polar of the point [Art 13, Cor 2

60 If a circle be drawn touching a conic at P and cutting it at Q and R, then will QR and the tangent at P be equally inclined to the axis Shew how to draw a circle touching a conic at any given point (other than a vertex) and also cutting it at the same point

The Parabola.

61. If the ordinates or focal distances of all points on a parabola be cut in a given ratio the locus of the points of section will in either case be a parabola

62 The perpendicular from P to a chord AP meets the axis at a distance equal to the latus rectum from the foot of the ordinate of P.

63 Circles whose radii are in arithmetical progression touch a given straight line on the same side at a given point If to each circle a tangent parallel to the given line be drawn, it will cut the circle next larger in points lying on a parabola

64 Prove the following construction. Take any ordinate NP, and draw PM equal and parallel to NA Divide NP into any number of equal parts, and through the points of section draw parallels p_1, p_2, p_3, to the axis Divide MP into the same number of equal parts in points 1, 2, 3, Then the lines p_1, p_2, p_3 meet $A1$, $A2$, $A3$, respectively on the parabola

65 Deduce from Art 19 that the middle points of all parallel chords of a parabola are at the same distance from the axis

66 A point on a parabola being given, if the focus also be given the envelope of the directrix will be a circle, or if the directrix be given the locus of the focus will be a circle

67 The directrix and one point being given, prove that the parabola will touch a fixed parabola to which the given straight line is tangent at the vertex.

68 Given the directrix of a parabola and two points on the curve, two positions of the focus can be determined, or given the focus and two points, two positions of the directrix can be determined.

69 If two parabolas have a common focus, their common chord passes through the intersection of their directrices and bisects the angle between them

70 If two parabolas have a common directrix, their common chord bisects the straight line joining their foci at right angles

71 Find the locus of the centre of a circle which passes through a given point and touches a given straight line, or which touches a given circle and a given straight line

72 Find the locus of a point which moves so that its shortest distance from a given circle is equal to its perpendicular distance from a given diameter of that circle

73 Determine the position of P so that the triangle SPG (Art 24) may be equilateral

74 If an equilateral triangle circumscribe a parabola, the focal radii to its vertices pass each through the opposite point of contact [Art 27

75 A focal chord being drawn to meet the tangent at a constant angle, determine the locus of their intersection

76 The circle on a chord of a parabola as diameter does not meet the directrix unless the chord passes through the focus

77 The circle described on any focal chord of a parabola as diameter touches the directrix, and the circle on any focal radius touches the tangent at the vertex

78 Circles being described on the segments of a focal chord as diameters, the straight line joining their centres subtends right angles at the intersections of their common tangents

79 Prove also that the squares of their common tangents vary as the length of the chord

80 A point within a parabola is nearer to the focus than to the directrix

81 If P be any point on a parabola whose focus is S, and PM be perpendicular to the directrix, prove that the line bisecting the angle SPM is the tangent at P, according to Euclid's definition of a tangent

82. In Art 20 shew that the angle QOQ' is equal to the angle MYM'. In what case are these angles right angles?

83 Prove the following construction for drawing tangents to a parabola from an external point T With centre T and radius TS describe a circle cutting the directrix in M and N, then the diameters through these points meet the curve in the points of contact of the tangents

84 Shew that all parabolas are similar curves

85 A parabola being given find its axis and focus

86 Shew how to place in a parabola a focal chord of given length

87 The perpendicular to a chord of a parabola from its middle point V meets the axis at a distance equal to SX from the foot of the ordinate of V

88 Shew that the locus of the middle point of a focal chord, or of any chord which meets the axis in a fixed point, is another parabola

89 If PQ be a focal chord of a parabola, $SA \; PQ = SP \; SQ$

90 The semi-latus rectum is a mean proportional between the principal ordinates of the ends of a focal chord And if AM, AM' be the corresponding abscissae, then $AM . AM' = AS^2$

91 If PQ be a focal chord, AP, AQ meet the latus rectum at distances from S equal to the ordinates of Q and P

92 If a leaf of a book be folded so that one corner moves along an opposite side the direction of the crease touches a parabola

93 The locus of the vertex of a parabola which has a given focus and a given tangent is a circle

94 A triangle revolves about its vertex in one plane prove that at any instant the directions of motion of all the points of its base are tangents to a parabola

95 The diameters through the extremities of any focal chord of a parabola meet the chords joining them to the vertex upon the directrix and intercept upon it a length which subtends a right angle at the focus

96 In Art 22 shew that $QD^2 = 4AS \; PV$

97 From a point O on the directrix of a parabola are drawn two tangents, and through the focus S two straight lines parallel

to these tangents, shew that the part of the directrix intercepted between these parallels is bisected in O

98 A circle can be described touching any two diameters of a parabola and the focal radii to their extremities

99. A chord QQ' is cut in O by a diameter which meets the curve in P Shew that if R be a point on the curve whose abscissa is PO, and PV, PV' be the abscissae of Q, Q', then

$$QV^2 - Q'V'^2 \,.\, QV^2 - OR^2 = QV + Q'V' \quad QV.$$

Deduce that $QV \; Q'V' = OR^2$ and $PV \; PV' = PO^2$

100 Any triangle whose base is parallel to the axis of a parabola has its remaining sides in the subduplicate ratio of the parallel focal chords [Art 30

101 If PQ be a chord which subtends a right angle at A, and AN, AM be the principal abscissae of P, Q, then PQ passes through the fixed point K in the axis, where $AK = 4AS$, and

$$AN.AM = PN \; QM = 16AS^2$$

102. A chord PQ of a parabola is a normal at P and subtends a right angle at the vertex prove that SQ is three times SP

103 If a circle cut a parabola in points 1, 2, 3 above the axis and in a point 4 below it, the difference of the ordinates of 1, 3 is to the difference of their abscissae as the sum of the ordinates of 2, 4 is to the difference of their abscissae Deduce that the ordinate of 4 is equal to the sum of the ordinates of 1, 2, 3

Examine the cases in which (1) 1, 2 coalesce, (2) 1, 2, 3 coalesce

104 If 1, 2 and 3, 4 lie on opposite sides of the axis the sum of the ordinates of 1, 2 is equal to the sum of the ordinates of 3, 4

105 If a circle and a parabola touch in one point and intersect in two others, the diameters of the parabola at the latter points will meet the circle again on a parallel to the tangent at the former

106. The tangents at P, Q meet in T, and O is the centre of the circle TPQ prove that OT subtends a right angle at S and that the circle OPQ passes through S [Art 27

107 If R be a point on a parabola and RS be produced to T so that $ST = SR$ and the tangents TP, TQ be drawn, prove that the circle TPQ touches the curve in R

108 Two parabolas which have a common focus and their axes in opposite directions intersect at right angles

109 Two given parabolas have the same focus and axis, and any line through the focus cuts them in P, Q, P', Q' shew that the tangents at these points form a rectangle one diagonal of which goes through the focus

110 Shew that the locus of intersection of tangents which are at right angles to two parabolas which have a common focus and axis is a straight line perpendicular to the axis Examine the case in which the directrices of the two parabolas coincide

111 If from any point T on a fixed tangent a second tangent TP be drawn, the angle STP will be constant Hence shew that if two fixed tangents be cut by any third in points p, q, the triangle Spq will have its angles constant

112 PQR being a circumscribed triangle, the perpendiculars from P, Q, R to SP, SQ, SR are concurrent

113 If one triangle can be inscribed in a given circle so that its three sides touch a given parabola, any number of triangles can be so described

114 Deduce from Art 29 that if the tangents at P, Q meet in T, the circle through P touching QT in T passes through the focus

115 Two fixed straight lines intersect in O prove that any circle through O and through another fixed point S meets the two fixed lines again in points such that the chord joining them touches a fixed parabola whose focus is S

116 The locus of the centre of the circle circumscribing the triangle formed by two fixed tangents to a parabola and any other tangent is a straight line

117 If two tangents to a parabola be cut by any third, their alternate segments have the same ratio, and this ratio is constant when the two tangents are fixed

118 If the two tangents from any point on the axis of a parabola be cut by any third tangent, their alternate segments will be equal

119 If T be the point of concourse of the tangents to a parabola at P and Q, and if p, q be the points in which any third tangent intersects them, then

$$\frac{Tp}{TP} + \frac{Tq}{TQ} = 1$$

120 Shew that the envelope of a straight line which is cut in a constant ratio by three fixed straight lines is a parabola touching the three fixed lines [Prob 111

121 The side PQ of a circumscribed triangle PQR meets the directrix in D, and RN drawn perpendicular to PQ meets SD in N; prove that N lies on the circle PQR Deduce that if a parabola be inscribed in a triangle, its directrix passes through the orthocentre

122 Shew that *one* parabola can in general be described touching four given straight lines.

123 Deduce from properties of the parabola the following geometrical theorems:—

(i) If from any point on the circumscribed circle of a triangle perpendiculars be let fall upon its three sides their feet will be collinear

(ii) The circumscribed circles of the four triangles formed by any four straight lines meet in a point

(iii) The orthocentres of the four triangles formed by any four straight lines are collinear

124 The loci of the centroid and the orthocentre of the triangle of Prob 116 are straight lines, and the locus of the centre of its nine-points circle is a straight line [Prob 120

125 If from the focus S of a parabola, SY, SZ be drawn perpendicular to the tangent and normal at any point, YZ will be parallel to the axis

126 The normals at the ends of a focal chord intersect at right angles upon its diameter, and the locus of their intersection is a parabola

127 The normal at any point is equal to twice the focal perpendicular upon the tangent, and is also a mean proportional between the focal distance of the point and the latus rectum

128 Two circles whose centres are on the axis of a parabola touch the parabola and one another Prove that the difference of their radii is equal to the latus rectum

129 Two points are taken on a parabola such that the sum of the parts of the normals intercepted between the points and the axis is equal to the part of the axis intercepted between the normals prove that the difference of the normals is equal to the latus rectum

130 The perpendicular SY being drawn to any tangent, a straight line is drawn through Y parallel to the axis to meet in Q the straight line through S parallel to the tangent prove that the locus of Q is a parabola

131 The normal at any point is equal to the ordinate which bisects the subnormal at that point.

132 The perpendicular to a normal to a parabola at the point in which the normal meets the axis envelopes an equal parabola, and the focal vector of the point at which the normal is drawn meets the envelope at the point in which the perpendicular touches it

133 The locus of the foot of the focal perpendicular on the normal is a parabola

134 The squares of the normals at the ends of a focal chord are together equal to the square of twice the normal perpendicular to the chord [Prob 127

135 The diameter through one end of a focal chord bisects the chord normal at the other

136 A normal chord of a parabola produced to meet the directrix subtends a right angle at the pole of the chord, and the polar of the middle point of the chord meets the focal radius to its point of concourse with the directrix upon the normal at its further extremity.

137 If T be the pole of a chord PQ normal at P, and AN be the abscissa of P, shew that

$$PQ \quad PT = PN \quad AN.$$

138 Prove also that the straight line drawn from S at right angles to ST bisects QT

139 If a parabola be made to roll upon an equal parabola, their vertices being initially coincident, the locus of the focus of the former will be the directrix of the latter.

140 The tangent at any point meets the directrix and the latus rectum in points equidistant from the focus

141 The vertex of a constant angle whose sides envelope a parabola traces a hyperbola having the same focus and directrix [Art 8

142 Tangents being drawn to a parabola from any point T, the diameters through their points of contact meet any secant PQ which passes through T in M and N shew that

$$TM^2 = TN^2 = TP \cdot TQ$$

143 Given two chords of a parabola, find the direction of its axis, and shew that there are two solutions

144 If a circle and a parabola touch and cut one another at the same point (Prob 60), their common chord is equal to four times their common tangent at that point, terminated by the axis

145 If R be any point on the tangent at P to a parabola and if the diameter through R meet the curve in Q, then will RP^2 vary as RQ

146 A diameter meeting a chord and the tangent at an end of it is cut by the curve in the ratio in which it cuts the chord

147 Draw a chord which shall be cut in a given ratio by a given diameter

148 The intercepts on any diameter of a parabola between any two tangents and the ordinates of their points of contact to that diameter are equal, and the triangle contained by the two tangents and the diameter is equal to half the trapezium bounded by their chord of contact, the two ordinates and the diameter

149. Three tangents to a parabola form a triangle equal to half the triangle determined by their points of contact

150 The area of a parabolic segment is to a triangle of the same base and altitude as four to three

The Central Conics.

151 In Art 33 shew that $Z'Ap$ and ZpA' are straight lines

152 The sides AD, DC of a rectangle $ABCD$ are divided into the same number of equal parts, and straight lines are drawn from B, A respectively to the points of section Shew that corresponding lines in the two series meet on an ellipse whose axes are equal to the sides of the rectangle

153 A parallelogram $ABCD$ has its diagonal AC at right angles to the side AB If CD be divided into any number of equal parts, and straight lines be drawn from A to the points of

section, and if AC be divided into the same number of equal parts and straight lines be drawn from B to the points of section, then will corresponding lines in the two series meet on a hyperbola

154 Given one focus of a central conic, a point on the curve, and the length of the axis, find the locus of the further focus, and the locus of the centre

155 If two ellipses whose major axes are equal have a common focus, they will intersect in two points only, and their common chord will be at right angles to the straight line joining their centres [Art 5

156 What is the locus of the centre of a circle which touches two fixed circles?

157 Given a central conic, find its centre and foci

158 Shew that the sum (or difference) of the focal distances of any point without the conic is greater than the transverse axis, and conversely

159 Draw a tangent to a conic parallel to a given line

160 A conic is drawn touching an ellipse at the extremities A, B of the axes, and passing through the centre C of the ellipse, prove that the tangent at C is parallel to AB

161 If the perpendicular from the centre on the tangent at P meet the focal distance SP produced in R, the locus of R is a circle whose diameter is equal to the transverse axis

162 Given a focus of an ellipse, the length of the transverse axis, and that the second focus lies on a straight line, prove that the ellipse will touch two fixed parabolas having the given focus for focus

163 The circle inscribed in the triangle SPS' touches SP in M, and SS' in N Prove that $PM = AS$, and $AN = SP$

164 From a point in the auxiliary circle straight lines are drawn touching the ellipse in P and P', prove that SP is parallel to $S'P'$

165 A diameter of an ellipse varies inversely as the perpendicular focal chord of the auxiliary circle [Art 46

166 If SY, SZ be perpendiculars on two tangents which meet in T, the line through T perpendicular to YZ will pass through S'

167 Given a focus S and two tangents, the locus of the second focus is a straight line

168 If SY, SZ be drawn perpendicular respectively to the tangent and normal at any point, YZ will pass through the centre.

169 The ordinates to the axes at the points in which a common diameter meets the major and minor auxiliary circles of an ellipse intersect two and two on the ellipse

170 A given point P in a given straight line AB which slides between two fixed straight lines at right angles traces an ellipse, whose semi-axes are equal to AP and BP [Art 35

171 Deduce the theorems of Art 43

172 A circle can be drawn through the foci and the intersection of any tangent with the tangents at the vertices

173 Any diameter is divided harmonically by a double ordinate and the point of concourse of the tangents at its extremities

174 The exterior angle between two tangents to an ellipse is an arithmetic mean to the angles which the chord of contact subtends at the two foci What is the corresponding theorem when the direction of the chord of contact falls between the foci?

175 The focal radii to the two ends of a diameter make equal angles with the tangents thereat

176 The ordinate PN bisects the angle YNY' (Art 39), and the points $YNCY'$ are concyclic

177 Also $SY^2 \;\; CB^2 = SP \;\; 2CA \pm SP$

178 Also, if CD be the radius conjugate to CP,

$$SY \;\; SP = CB \;\; CD$$

179 The normal at P is a harmonic mean to SY, $S'Y'$ (Art 39), and is bisected by $S'Y$ and by SY'

180 Tangents being drawn from any point on a circle through the foci, shew that the bisectors of the angles between them pass through fixed points [Art 40, Cor 1

181 If the tangent and normal meet either axis in T, G, then $CG \;\; CT = CS^2$

182 If P be any point on a conic whose foci are S and S', the circles on SP, $S'P$ as diameter touch the auxiliary circle and have for their radical axis the ordinate of P

183 The pole of the tangent at P with respect to the auxiliary circle lies on the ordinate of P

184 A circumscribing parallelogram which has two corners on the directrices has the other two on the auxiliary circle [Art 7, Cor

185 Prove that if one rectangle can be inscribed in a given circle so that its sides touch a given conic, any number of rectangles can be so described

186. If an ellipse inscribed in a triangle has one focus at the orthocentre, the other focus will be at the centre of the circumscribing circle

187 Prove also that the transverse axis of the ellipse is equal to the radius of the nine-points circle of the triangle, and that the ellipse has double contact with the circle

188 If an ellipse slide between two straight lines at right angles the locus of its centre is a circle

189 The straight line joining the foci subtends at the pole of a chord half the sum or difference of the angles which it subtends at the extremities of the chord

190 The portion of a normal chord intercepted between the directrices subtends at the pole of the chord half the sum of the angles which the straight line joining the foci subtends at the extremities of the chord [Prob 44.

191 The pole of any straight line with respect to a central conic may be found by joining the points in which it meets the directrices to the nearer foci, and drawing perpendiculars through the latter to the joining lines

192 Every ellipse has one pair of equal conjugate diameters, and they coincide with the diagonals of the rectangle formed by the tangents at the extremities of the axes Has the hyperbola any corresponding property?

193 If CP and CD be conjugate radii of an ellipse,

$$(SP - CA)^2 + (CA - SD)^2 = CS^2$$

194 When is the sum or difference of conjugate diameters greatest, and when least?

195 The tangent from N to the circle on XX' (Art 49) as diameter varies as the normal at P, and the tangent to the auxiliary circle varies as PN

196 If N be a point in AA' produced the circles described about S, S' with radii AN, NA' meet on the hyperbola What is the corresponding construction for the ellipse?

197 From a fixed point O, OP is drawn to a given circle Find the envelope of a straight line through P inclined at a constant angle to OP

198 In a central conic a circle through P and either G or g cuts off from the focal distances lengths whose sum is constant

199 Given in an ellipse a focus and two points, the other focus describes a hyperbola

* 200 TP, TQ are the tangents from T, prove that a circle can be described with T as centre so as to touch SP, HP, SQ, and HQ, or these lines produced What does this become for the parabola?

201 If P, Q be points on a central conic, a confocal passes through the intersections of SP, $S'Q$ and SQ, $S'P$, and the tangents at these points and at P, Q cointersect

202 If PP', DD' be conjugate diameters of a hyperbola and Q any point on the curve, shew that $QP^2 + QP'^2$ exceeds $QD^2 + QD'^2$ by a constant quantity

203 Given two points of a parabola and the direction of its axis, the locus of the focus is a hyperbola

204 A chord which subtends a right angle at the vertex meets the axis in a fixed point

205 P being any point on an ellipse, the locus of the centre of the circle inscribed in the triangle SPS' is an ellipse. What is the locus of the centre of the circle touching the transverse axis of an ellipse, SP, and $S'P$ produced?

206 In a hyperbola the locus of the centre of the circle inscribed in the triangle SPS' is a straight line, and the locus of the centre of the circle touching the transverse axis, SP and $S'P$ produced, is a hyperbola

207 If a hyperbola touches the sides of a quadrilateral inscribed in a circle and if one focus lies on the circle, the other lies on the circle

208 The triangle whose base is equal to the transverse axis, and its remaining sides to the focal distances of any external point, has its vertical angle equal to the angle between the tangents to the conic from that point and its remaining angles to the angles which either tangent subtends at the foci

[Arts 39, 40, Cor 1

209 The projection of the normal at any point, terminated by the conjugate axis, upon either focal distance is equal to the semi-axis transverse

210 The focal distances of g (Art 42) meet the directrices upon the parallel to the axis through P

211 If AM and $A'M'$ be taken on the axis equal to the focal distances of any point P on an ellipse, then

$$CP^2 = CB^2 + CM^2$$

Deduce the property of the principal ordinate

212 If two ellipses having equal axes be placed vertex to vertex, and one of them roll upon the other, either of its foci will describe a circle about a focus of the latter

213 The common diameters of equal, similar and concentric ellipses are at right angles [Art 14, Cor 1

214 The diagonals of any parallelogram circumscribing a conic are conjugate diameters, and the sides of any inscribed parallelogram are parallel to conjugate diameters

215 The sum or difference of the reciprocals of the squares of any two diameters at right angles is constant

216 The inscribed parallelogram whose diameters are at right angles envelopes a circle, the reciprocal of the square of whose radius is equal to $\frac{1}{CA^2} \pm \frac{1}{CB^2}$

217. Determine the positions of a chord of an ellipse which subtends right angles at both foci [Art 42, Cor

218 The opposite sides of a quadrilateral described about an ellipse subtend supplementary angles at either focus

219 If a triangle ABC circumscribe a conic the sum of the angles subtended by BC at the foci will exceed the angle A by two right angles

220 An ellipse touches the sides of a triangle, prove that if one of its foci move along the arc of a circle passing through two of the angular points of the triangle, the other will move along the arc of a circle passing through the same two angular points

221 A circumscribed quadrilateral whose diagonals meet at the centre of the conic must be a parallelogram

222 If P and Q be points on a conic, CM and CN their abscissæ, and T the point in which PQ meets the axis, then

$$CT(PM-QN)=PM\ CN-QN\ CM\ *$$

223 If CP, CD and CP', CD' be conjugate radii, and if PN, DR be ordinates to CP', then

$$CN^2 \pm CR^2 = CP'^2, \quad PN^2 \pm DR^2 = CD'^2,$$

and
$$PN\ CR = DR\ CN = CD'\ CP'$$

224 The vertices of the conjugate parallelograms of an ellipse lie on a similar ellipse, and their polars envelope a similar ellipse What are the corresponding properties of the hyperbola?

225 The parallelograms whose diagonals are any two diameters and their conjugates respectively are equal

226 With the orthocentre of a triangle as centre two ellipses are described, the one touching its sides and the other passing through its vertices prove that they are similar, and that their homologous axes are at right angles [Art 46

227 If two ellipses having equal major axes be inscribed in a parallelogram, their foci determine an equiangular parallelogram

228 Any circle through the focus S and the further vertex A' of a hyperbola whose eccentricity is two, meets the curve in three points P, Q, R which determine an equilateral triangle, and conversely the circumscribing circle of any equilateral triangle inscribed in a hyperbola whose eccentricity is two, passes through a focus and the further vertex.

* Equate the areas $(CPT-CQT)$ and $(CPM+PMNQ-QCN)$.

229 Any one of a series of conterminous circular arcs may be trisected by drawing a pair of hyperbolas whose eccentricity is two, and whose centres and vertices trisect the chord of the arc How does it appear from this construction that the problem, to trisect a given angle, admits of three solutions?

230 Prove that in Prob 228 the focal radii SP, SQ, SR meet the curve in three other points which determine an equilateral triangle, and shew that the triangle PQR envelopes a fixed parabola having S and the S-directrix for focus and directrix

231 Draw a pair of conjugate diameters inclined at a given angle, and thence determine the axes and foci

232 If two points E and E' be taken in the normal PG to an ellipse such that $PE = PE' = CD$, where CD is the radius conjugate to CP, the loci of E and E' are circles, whose diameters are equal to the sum and difference of the axes of the ellipse

233 Prove also that the axes bisect the angles between the lines CE, CE' Deduce a construction for determining the axes of an ellipse when two conjugate diameters are given.

234 For a hyperbola, the loci of E and E' are hyperbolas having their axes equal to the sum and difference of the axes of the given hyperbola

235 The tangent at P meets any two conjugate diameters in T, t and TS, tH meet in Q, prove that the triangles SPT, HPt, TQt are similar, and also that the area of the triangle CPT varies inversely as that of CPt

236 If the tangent at Q (Fig. Art 41) meet two parallel tangents in R and R', then will the radius parallel to the tangent be a mean proportional to QR and QR' Shew also that the radius parallel to RP is a mean proportional to PR, $P'R'$

[Art 47

237 The common tangents to an ellipse and to a circle through the foci touch the circle in points lying on the tangents at the ends of the minor axis

238 If any two points P, Q be given on a conic, prove that a third point R may be found so that the angle PRQ is a maximum by the following construction

Draw a tangent parallel to PQ, touching the ellipse in K, and

draw KR perpendicular to the major axis, cutting the curve again in R*

239 The two points on a central conic at which any chord subtends the greatest and least angles are at the ends of a diameter equal to that which bisects the chord

240 If two chords be drawn from any point of a conic equally inclined to the normal at that point, the tangents at their further extremities will intersect upon the normal †

241 Supplemental chords of a conic which are equally inclined to the curve at their common point have their poles upon the orthocycle, and their sum or difference is equal to the diameter of the same ‡

242 A bifocal conic being defined as the locus of a point P the sum or difference of whose distances from two fixed points S, S' is constant, prove, by taking Euclid's definition, that the tangent is the bisector (external or internal) of the angle SPS', and prove also the property of the directrices and the property of the principal ordinate§

243 Prove that two confocal conics of the same species do not meet‖

244 If from any point of a conic tangents are drawn to a confocal conic, these tangents are equally inclined to the normal at the point [Art 40, Cor 1

245 The bisectors of the angles between the tangents from any point are tangent and normal to the confocals through that point Prove that confocal conics cut at right angles

* The chord PQ must subtend *equal* angles at R, and a consecutive point on the curve Hence R lies on one of the segments of circles described upon the chord so as to touch the conic.

† If PQ, PQ' be two such chords and the normal PP' and the tangent at P meet QQ' in K, T respectively, $Q'KQT$ is a harmonic range Then see Prob 59

‡ Here the normal PP' is an ordinate of QQ', and is parallel to the tangent at Q

§ If P' be any point on the bisector of the angle SPS' then, for the ellipse, $SP'+S'P'>SP\dotplus S'P$, and similarly for the hyperbola Also if in the ellipse $SP\dotplus S'P=2CA$ and CN be the abscissa, it may be shewn that

$$SP^2 \sim S'P^2=4CS\ \ CN \text{ and } SP\sim S'P=2e\ \ CN,$$

where e stands for the ratio $CS\ \ CA$ It may also be shewn (Lemma D) that

$$(SP\dotplus S'P)^2+(SP\sim S'P)^2=4\,(CS^2+CN^2\dotplus PN^2)$$

Similar remarks apply to the hyperbola

‖ Conics which have the same foci are called *confocal* conics

246 Draw figures illustrating a system of confocal conics, shewing that special cases are,—a circle, a straight line perpendicular to the major axis, the line joining the foci, and its complement* To what does the theorem that confocal conics cut at right angles reduce when the two foci coalesce?

247 Shew that if the sides of a rectangle touch two confocal conics, its vertex lies on a fixed concentric circle

248 Shew that for any two confocal conics, the difference of the squares of the distances from the centre of parallel tangents is constant

249 If a circle be drawn through the foci of two confocal conics, cutting them in P, Q, the tangents at P, Q will intersect on the circumference of the circle

250 If a chord of a central conic be produced to meet the directrices, the parts produced will subtend equal angles at the pole of the chord

The Asymptotes.

251 Given the asymptotes of a hyperbola and a point on the curve, determine the foci and directrices

252 The perpendicular from a focus to an asymptote meets it upon the corresponding directrix, and the point of intersection lies on the auxiliary circle Shew also that the point of contact of the tangent from such a point of intersection lies on a focal radius parallel to the asymptote

253 At any point P of a hyperbola SP is equal to a line drawn parallel to an asymptote to meet the directrix.

254 If the tangent at any point P cut an asymptote in T, and SP cut the same asymptote in Q, then $SQ = QT$

255 Shew that the asymptotes of a hyperbola may be regarded as tangents whose points of contact are at infinity, and deduce that the lines from the foci to any point on the curve make with one of the asymptotes a triangle of constant perimeter [Art 9

256 The line joining a pair of adjacent extremities of any two conjugate diameters is parallel to one asymptote and is bisected by the other

* The remainder of an unlimited straight line from which any part has been taken away is called the *complement* of that part

257 The asymptotes and any two conjugate diameters divide any straight line harmonically *

258 Supposing the axes of a hyperbola to vanish whilst its eccentricity remains unaltered, determine the limiting form of the curve †

259 Given a pair of conjugate diameters, two hyperbolas can be described these have the same asymptotes, and every two diameters conjugate in the one are conjugate in the other ‡

260. If two conjugate diameters of a hyperbola be equal, every two conjugate diameters must be equal and the asymptotes must be at right angles

261 If two hyperbolas have the same asymptotes a chord of one touching the other is bisected at the point of contact

262 If tangents be drawn to a hyperbola from any point on the conjugate hyperbola, their chord of contact will touch the opposite branch of the latter and be bisected at its point of contact

263 The tangent to a hyperbola at P meets an asymptote in T and TQ is drawn parallel to the other to meet the curve in Q, prove that if PQ meet the asymptotes in L and M, the line LM will be trisected at P and Q

264 A hyperbola can be drawn through the ends of any two radii of an ellipse so as to have the conjugate diameters as asymptotes

265 If through two points R, R' of a hyperbola lines be drawn parallel to the asymptotes forming the parallelogram $RTR'T'$, shew that TT' goes through the centre C, and that $CT \cdot CT' = CP^2$, where P is a point where CTT' cuts the curve

266 If the abscissæ on either asymptote of any number of points on a hyperbola are in arithmetical progression, their ordinates are in harmonical progression

267 The straight lines joining the points in which any two tangents to a hyperbola meet the asymptotes are parallel, and the intercepts which the tangents make upon the asymptotes are bisected by their chord of contact

* Prove that they divide the tangent at a vertex harmonically The theorem is really contained in Prob 256 See Lemma G

† Here the foci and directrices coalesce into the centre and the line along which the minor axis was measured

‡ Two hyperbolas thus related are said to be *conjugate*

268 Find the locus of a point which divides the part of any tangent intercepted between the asymptotes in a constant ratio

269 Find the locus of the centroid of a triangle of constant area contained by one variable and two fixed straight lines

270 If the ordinate at P to either axis meets the nearer asymptote in E, the perpendicular through E to the asymptote passes through the point in which the normal at P meets that axis

271 The four normals to a hyperbola and its conjugate at points lying upon a perpendicular to either axis meet one another upon that axis

272 Any tangent and its normal meet the asymptotes and the axes respectively in four points which lie on a circle passing through the centre of the curve, and the radius of this circle varies inversely as the perpendicular from the centre on the tangent

273 The intercept on any tangent between the asymptotes subtends at the further focus an angle equal to half the angle between them it also subtends a constant angle at the intersection of its normal with either axis

274 The chords of intersection of any circle with the asymptotes are equally inclined to the axis Shew that this agrees with Prob 258

275 The products of the segments of any two intersecting chords of the asymptotes are as the parallel focal chords of the hyperbola

276 Any circle which touches both branches makes an intercept equal to the transverse axis on either asymptote

277 The lines joining a variable point on a hyperbola to two fixed points on it intercept a constant length on either asymptote

278 The axis of the two parabolas which have a common focus and pass through two given points are parallel to the asymptotes of the hyperbola which passes through their focus and has the given points for foci [Prob 68

279 If an ellipse and a confocal hyperbola intersect in P, an asymptote passes through the point on the auxiliary circle of the ellipse which corresponds to P

280 The area of the hyperbolic sector determined by any two radii CP, CQ is equal to the area cut off from the space between the asymptotes by the parallels from P and Q to either asymptote

The Equilateral Hyperbola.

281 The subnormal at any point of an equilateral hyperbola is equal to the central abscissa, the tangent from the foot of the ordinate to the auxiliary circle is equal to the ordinate, and the projection of the normal (terminated by either axis) upon either focal vector is equal to the semi-axis

282 The centre of an equilateral hyperbola circumscribing an equilateral triangle lies on the inscribed circle of the triangle, and the centre of the circle lies on the hyperbola

283 The locus of the middle point of a straight line which cuts off a constant area from a corner of a square is an arc of a rectangular hyperbola

284 CY being drawn perpendicular to the tangent at a point P of an equilateral hyperbola, the angle PCY is bisected by the transverse axis, and the triangles PCA, CAY are similar

285 If CP, CD be conjugate radii of a rectangular hyperbola, then will D be the reflexion of P with respect to one of the asymptotes.

286 Any two conjugate semi-diameters contain equal and similar triangles with the ordinates and abscissæ of their extremities to any other diameter

287 The ends of the equal conjugate diameters of a series of confocal ellipses lie on the confocal equilateral hyperbola.

288 If PQ and $P'Q$ be any pair of supplemental chords of a rectangular hyperbola, the bisectors of the angle PQP' are parallel to the asymptotes, and if the tangent at Q and its ordinate to PP' meet that diameter in T and V, then CP and TP' subtend equal angles at Q, and the circle CQT touches QV

289 Diameters at right angles bisect chords at right angles, and any chord subtends equal or supplementary angles at the ends of a perpendicular chord

290 Of two chords at right angles or conjugate in direction, one and one only is a chord of a single branch

291 On opposite sides of a chord of a rectangular hyperbola equal segments of circles are described Shew that the four points in which the completed circles meet the curve again are the vertices of a parallelogram

292 Tangents to a parabola which include the supplement of half a right angle intersect on an equilateral hyperbola [Art 8

293. Every right-angled triangle inscribed in an equilateral hyperbola has its hypotenuse parallel to the normal at the opposite angle Hence shew how to draw a tangent at any given point on the curve [Art 56

294 The base of a triangle and the sum or difference of its base angles being given, the locus of its vertex is an equilateral conic Determine the asymptotes of the hyperbola by supposing the vertex of the triangle to be infinitely distant [Art 54

295 The chords connecting the ends of a fixed diameter of a circle and of any double ordinate of the same intersect upon an equilateral hyperbola

296 If on an arc AB of a circle whose centre is O there be taken two points P, Q such that arc $AP = 2$ arc BQ, then a rectangular hyperbola described on AO as diameter so as to pass through the intersection of OB with the tangent to the circle at A will also pass through the intersection of AP, OQ

297 Shew also that the hyperbola and the completed circle intersect in three points (other than A) which determine an equilateral triangle, and deduce that the problem, to trisect a given angle, admits of three solutions

298 The opposite arcs cut off by any two diameters subtend equal angles at any point on the curve

299 If two concentric rectangular hyperbolas are such that the axes of one are the asymptotes of the other, they cut each other at right angles, and any common tangent subtends a right angle at the centre

300. A circle through the centre and two points of a rectangular hyperbola passes also through the intersection of the

lines drawn from each of the two points parallel to the polar of the other

301 Ellipses being inscribed in a parallelogram, their foci lie on an equilateral hyperbola [Prob 294

302 A conic through the centres of the four circles which touch the sides of a triangle is a rectangular hyperbola, and its centre is on the circumscribing circle [Art 56

303 A conic through the four common points of two rectangular hyperbolas is itself a rectangular hyperbola

304 The tangents to an equilateral hyperbola at the vertices of an inscribed triangle meet two and two on the lines joining the feet of the perpendiculars of the triangle [Prob 59

305 If each vertex of a triangle be the pole of the opposite side with respect to an equilateral hyperbola, the circumscribing circle will pass through the centre of the hyperbola*

306 A circle and an equilateral hyperbola intersect in four points: if one of their common chords is a diameter of the hyperbola, the other is a diameter of the circle, and the tangents to the circle at the ends of this diameter are ordinates of the diameter of the hyperbola

307 The circles described upon the six common chords of any two equilateral hyperbolas as diameters cut one another orthogonally in opposite pairs

308 The circles described on parallel chords of an equilateral hyperbola as diameters have a common radical axis

309 A circle meets an equilateral hyperbola in four points O, P, Q, R and OO', PP', QQ', RR' are diameters of the hyperbola: prove that O' is the orthocentre of the triangle PQR, and similarly for the others

310 Two circles touch the same branch of an equilateral hyperbola and touch each other in the centre: prove that the chord of the hyperbola joining the points of contact subtends at the centre an angle equal to the angle of an equilateral triangle

* A triangle each of whose vertices is the pole of the opposite side with respect to a conic is called a *self-conjugate* or *self-polar* triangle

The Cone.

311 In Art 59 shew that $AS = A'H$, and $SH = OA \pm OA'$

312 A conic section may be regarded as the locus of a point the sum or difference of whose distances from a point in its plane and a point without it is constant

313 Express the eccentricity of a section of a cone in terms of the angles which the axis of the cone makes with its sides and with the axis of the section*

314 The sections by identical planes of the cones touching two given spheres have their eccentricities in a constant ratio

315 All sections of a right cone made by planes parallel to tangent planes of the cone are parabolas, and their foci lie on a cone having with the first a common vertex and axis

316 If two or more plane sections have the same directrix, the corresponding foci lie on a straight line through the vertex of the cone

317 The sphere having for diameter the line joining the centres of the focal spheres of a section contains its auxiliary circle

318 The perpendiculars upon any tangent to a section from the centres of its focal spheres are at right angles to one another Deduce that the product of the focal perpendiculars upon the tangent is constant

319 If T be any point on the tangent at P to a section, the two tangents to it from T are inclined at the same angle as the tangents TQ, TR [Art 59 (ii)] to the spheres Deduce the property of the orthocycle [Art 59, Cor

320 The conjugate axis of a section is a mean proportional to the diameters of its focal spheres

321 The latus rectum of any section whose plane touches a sphere about the vertex of the cone as centre is equal to the diameter of the circular sections whose planes touch the sphere.

* One form of the value of the eccentricity is $\cos\alpha \sec\beta$, where α and β are the inclinations of the axis of the cone to the axis of the section, and to a side of the cone respectively

322 The sections of a cone by parallel planes are similar curves, and the asymptotes of the hyperbolic sections made by parallel planes are parallel to the sides of the cone which lie on the parallel plane through its vertex

323 Shew how to cut a section of maximum eccentricity from a given cone

324 A conic section may be regarded as the locus of a point the sum or difference of the tangents from which to two fixed circles is constant

325 The vertex of a right circular cone which contains a given ellipse lies on a certain hyperbola, its axis touches the hyperbola, and the sum or difference of the distances of a current point on the ellipse from any two points taken on opposite branches or the same branch respectively of the hyperbola is constant [Prob 312

Projection.

326 Prove by the method of projection that tangents to an ellipse at the extremities of any chord intersect on the diameter which bisects the chord

327 Deduce from a known property of the circle that the area of the conjugate parallelogram of an ellipse is constant

328 TP, TQ are tangents to an ellipse, and CP', CQ' are the parallel radii, prove that PQ is parallel to $P'Q'$

329 Any two similar and coaxal ellipses may be projected into concentric circles Hence shew that a chord of an ellipse which always touches a similar and coaxal ellipse is bisected at its point of contact, and that it cuts off a constant area from the outer ellipse and shew that the portions of any chord intercepted between the two curves are equal.

330 Find the locus of the point of intersection of the tangents at the extremities of pairs of conjugate diameters of an ellipse

331 Find the locus of the middle point of the lines joining the extremities of conjugate diameters

332 Any two radii of a circle and a pair of radii at right angles thereto determine equal triangles what is the corresponding property of the ellipse?

333 The orthogonal projection of a parabola is a parabola, and an ellipse or hyperbola may be projected into an ellipse or hyperbola of any eccentricity

334 If a chord of an ellipse and the tangent at its extremities contain a constant area, the chord cuts off a constant area from the ellipse and touches a similar ellipse, and the tangents at its extremities intersect on another similar ellipse

335 A polygon described about an ellipse so as to have its sides bisected at their points of contact is of constant area, and the polygon formed by joining every two successive points of contact is of constant area

336 Any double ordinate to a given diameter of an ellipse being divided into segments whose product is constant, the point of section traces a similar coaxal ellipse

337 Prove that the greatest triangle which can be inscribed in an ellipse is that which has its sides parallel to the tangents at its angular points and its centroid at the centre of the ellipse

338 The least triangle circumscribing a given ellipse has its sides bisected at the points of contact

339 The greatest ellipse which can be inscribed in a given parallelogram is that which bisects its sides

340 If a triangle be inscribed in an ellipse, the parallels through its vertices to the diameters bisecting the opposite sides meet in a point

341 Parallel chords drawn to an ellipse through the extremities of conjugate diameters meet the curve again at the extremities of conjugate diameters

342 Through the centre of an ellipse and the points of concourse and contact of any two tangents a similar and similarly situated ellipse can be drawn

343 Through a given internal point draw a straight line cutting off a minimum area from a given ellipse

344 If the tangent at the vertex A of an ellipse meets a similar coaxal ellipse in T and T', any chord of the former drawn from A is equal to half the sum or difference of the parallel chords of the latter through T and T'

345 The tangents to an ellipse at P and P' are parallel, any two conjugate diameters meet them in D and D', and any third tangent meets them in T and T', shew that

$$PD \cdot PT = P'T' \cdot P'D'$$

346 A triangle ABC inscribed in an ellipse has its centroid at the centre of the ellipse; shew that the tangents at the opposite extremities of the diameters through A, B, C form a triangle similar to and four times as great as the triangle ABC.

347 If two conjugate hyperbolas having a pair of conjugate diameters of an ellipse for asymptotes cut the ellipse at points lying on four diameters 1, 2, 3, 4 taken in order then will 1, 3 and 2, 4 be conjugate in the ellipse, and 1, 4 and 2, 3 in the hyperbolas

348 The locus of the middle point of a chord of an ellipse drawn through a fixed point is a similar ellipse, having its centre midway between the fixed point and the centre of the given ellipse

349 Given the directions of two sides of a triangle inscribed in a given ellipse, determine the envelope of its third side

350 Prove that the perpendiculars from any point on a circle to a fixed chord and to the tangents at its extremities are continued proportionals What is the corresponding property of the ellipse?

Curvature.

351 The radius of curvature at an extremity of the latus rectum of a parabola is equal to twice the normal

352 The diameter at either extremity of the latus rectum of a parabola passes through the centre of curvature at its other extremity

353 Determine the position of the common chord of a parabola and its circle of curvature at an extremity of the latus rectum

354 The circle of curvature at a point P of a conic cuts off from the diameter through P a portion equal to the parameter of that diameter*.

355 If the tangent at any point P of a parabola meet the axis in T, and if the circle of curvature meet the curve in Q, then $PQ = 4PT$.

* The *parameter* of any diameter of a central conic is defined as a third proportional to that diameter and its conjugate

356 At any point P of a parabola, if PY be the projection of SP upon the tangent, the chord of curvature through the vertex is a third proportional to AP and $2PY$

357 If R be the middle point of the radius of curvature at P in a parabola, PR subtends a right angle at S

358 The radius of curvature at any point of a parabola is double the portion of the normal intercepted between the curve and the directrix

359 Shew that the centre of curvature may be regarded as the point of ultimate intersection of two consecutive normals to the conic

360 If Q and Q' are points on a parabola on the same side of the axis and V the middle point of QQ', shew that the ordinate of the point of concourse of the normals at Q and Q' is to the ordinate of V as the product of the ordinates of Q and Q' to the square of the semi-latus rectum Hence determine the ordinate of the centre of curvature at P and the length of the radius of curvature

361 The tangent from any point of a parabola to the circle of curvature at its vertex is equal to the abscissa of the point

362 The envelope of the common chords of a parabola and its circles of curvature is a parabola, and the locus of their middle points is a parabola

363 If P, P', P'' be points on a parabola, P, P' on one side of the axis, and P'' on the other side, and the normals at P, P', P'' cointersect, prove that the sum of the ordinates of P and P' is equal to the ordinate of P''*

364 If from the vertex of a parabola chords AR and AR' be drawn equally inclined to the axis, the normals at the extremities of any chord parallel to AR intersect upon the normal at R', and the centre of curvature at the extremity of the diameter which bisects AR lies upon the normal at R'

365 A circle through the vertex of a parabola cuts the curve in general in three other points, the normals at which cointersect Prove also that the centroid of the triangle formed by the three points lies on the axis [Prob 103

* If the normals cointersect at a point whose projection on the axis is Z, we get $PN\ ZG=P'N'\ ZG'=P''N''\ ZG''$, whence, after some reduction, $PN+P'N'=P''N''$

366 Prove that at the vertex A of a conic the radius of curvature is equal to $AS\,(1+e)$, where e is the eccentricity, and at the vertex B to $\dfrac{CA^2}{CB}$

367 If the osculating circle at a vertex of an ellipse passes through the further focus, determine the eccentricity*

368 The circle through the foci of an ellipse and the extremity B of its minor axis will cut the minor axis in the centre of curvature at B

369 An ellipse, a parabola, and a hyperbola have the same vertex and the same focus shew that the curvature at the vertex of the parabola is greater than that of the hyperbola and less than that of the ellipse †

370 If P be a point of an ellipse equidistant from the minor axis and a directrix, the circle of curvature at P will pass through one of the foci

371 The tangent at P in an ellipse meets the axes in T and t, and CP is produced to meet the circle TCt in L prove that $2PL$ is equal to the central chord of curvature at P, and that $CL \cdot CP$ is constant

372 The circle of curvature at an extremity of one of the equal conjugate diameters of an ellipse passes through its other extremity

373 Find the points on a central conic at which the diameter of curvature is a mean proportional to the axes

374 At any point P of a rectangular hyperbola, the radius of curvature varies as CP^3

375 At any point of a rectangular hyperbola the diameter of the curve is equal to the central chord of curvature

376 At any point P of a rectangular hyperbola if CP be produced to Q, so that $PQ = CP$, and QO be drawn perpendicular to CQ to meet the normal at P in O, then O is the centre of curvature at P

377 At any point of a rectangular hyperbola the normal chord is equal to the diameter of curvature‡

* The circle of curvature at a point P on a conic is the circle of closest contact with the conic at P, and is called its *Osculating Circle* at that point

† *Curvature* is measured by the reciprocal of the radius of curvature

‡ This may be proved from properties of the centroid, orthocentre and centre of circumscribing circle of a triangle by taking three near points on the curve ultimately coincident and using Art 56

378 At any point P of a rectangular hyperbola, if PN be perpendicular to an asymptote, the chord of curvature in the direction PN is equal to $\frac{CP^2}{PN}$

379 From the point in which the tangent to an ellipse at P meets the major axis a straight line is drawn bisecting one of the focal distances and meeting the other in Q Prove that PQ is one-fourth of the focal chord of curvature at P

380 A hyperbola which touches an ellipse, and has a pair of its conjugate diameters for asymptotes has the same curvature as the ellipse at their points of contact

381 At a point P of an ellipse the chord of curvature in the direction of the ordinate PM is to PM as $2CD^2$ is to BC^2

382 In a central conic let the diameter CD parallel to the tangent at P meet PQ, the common chord of the ellipse and the circle of curvature at P, in K, then will PQ PK be equal to $2CD^2$, and the like is true for *any* chord of curvature PQ'

383 The normal chord which divides an ellipse most unequally is a diameter of curvature, and is inclined at half a right angle to the axis*

384 Prove that if two straight lines make *supplementary* angles with any third straight line, their projections make supplementary angles with the projection of that third line.

Hence, or otherwise, prove that if the circles of curvature at the extremities of two conjugate radii CP and CD of an ellipse meet the curve again in Q and R, PR is parallel to DQ

385 Given a point O on a circle, three positions may be found on the curve of a point P such that OP and the tangent at P make supplementary angles with a given diameter, and the three positions of P determine an equilateral triangle

Deduce by projection that there are three points on an ellipse, lying at the vertices of a maximum inscribed triangle, whose osculating circles cointersect at a given point on the ellipse

Prove also that the normals at the three points cointersect, and that the four points lie on a circle† [Prob 337

* The normal in two consecutive positions must cut off equal areas, and must be bisected at the centre of curvature

† If ABC be a maximum inscribed triangle, and the osculating circle at A meet the curve again in A', the tangent at A is parallel to BC, which is therefore equally inclined to the axis with AA'

Miscellaneous.

386. If PQ be a chord of a conic, and if the parallel focal chord F meet the tangent at P in T, then

$$PQ\ \ ST = F\ \ SP$$

387. Given the focus of a conic inscribed in a triangle, find the points of contact

388 If two chords AB, CD of a conic (not being parallel to one another) make equal angles with the axis, then will AC, BD and likewise AD, BC make equal angles with the axis

389 If a chord of a conic subtends equal angles at the extremities of another chord, it subtends equal angles at the extremities of any chord parallel to the latter

390 The tangent to a conic at a given point meets any two parallel tangents in points whose focal distances meet on a fixed circle, having its centre on the normal at the given point

391 If two focal chords of a hyperbola be conjugate in direction, the lines joining their extremities meet on the asymptotes, and in the equilateral hyperbola pass through fixed points on the asymptotes

392 One triangle being inscribed and another circumscribed to a parabola, if their sides be parallel each to each they will be in the ratio of four to one

393 If a parabola be inscribed in a given triangle, each chord of contact passes through a fixed point which lies on the bisector of the corresponding side of the triangle

394 If TP, TQ be tangents to an ellipse, and CP', CQ' the parallel radii, shew that the triangles TPQ and $CP'Q'$ are together equal to the trapezium $CPTQ$, and likewise to the triangle of Prob 208

Prove also that $TP\ TQ + CP'. CQ' = TS . TH$*

395 In Art 6 shew that to every two opposite positions of P at infinity corresponds a single point p on the directrix, and deduce that the hyperbola is to be regarded as touching its asymptotes at their extremities

Deduce also that all points at infinity in a plane may be regarded as lying on a straight line†, that this line is a tangent to every parabola in the plane, and that it cuts every real

* We have to shew that the triangle of Prob 208 is equal to $PTQ + P'CQ'$, which follows from Prob 225, taking into account that $PCQ = \frac{1}{2}(PSQ + PHQ)$

† First prove that in general a straight line corresponds to a straight line The conception of the line Infinity may also be arrived at by Conical Projection

hyperbola or ellipse in the plane in two real or imaginary points respectively

396 Deduce from Art 37 and Prob 170 two methods of describing an ellipse mechanically

397 If a straight rod $S'L$ be moveable in one plane about the end S', and a string LPS, fastened at L and another fixed point S, be stretched in contact with the rod by a pencil P, then the pencil will trace one branch of a hyperbola whose foci are S and S' How may the other branch be traced? Deduce a method of describing a parabola mechanically

398 If ABC be a triangle whose sides touch a conic at the points a, b, c, then $Ab \cdot Bc \cdot Ca = Ac \cdot Ba \cdot Cb$.

What is the corresponding theorem when the conic cuts the sides of the triangle? [Art. 16.

399 A central conic which passes through four given points has a pair of conjugate diameters parallel to the axes of the two parabolas which can be drawn through the same four points*

400 Prove that one conic can be drawn through five given points†, and that no two conics can intersect in more than four points

401 Prove by the parallelogram of forces that the locus of the centre of a conic touching four given straight lines is a straight line

* Let TP, TQ be tangents to an ellipse, and OAB, OCD chords parallel to them Determine a diameter of each of the two parabolas through A, B, C, D (Prob 143), then PQ and the diameter through T in the ellipse are parallel to the diameters of the parabola

† If A, B, C, D, E be the five points, let AC, BE meet in F, and draw from D a parallel to CA to meet BE in G, this meets the required conic in a point H given by $DG \cdot GH : BG . GE = AF \cdot FC : BF \cdot FE$ (Art 16) A diameter bisecting the parallel chords can then be drawn and in the same manner a second diameter

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Many of the above Problems are taken from the larger work named in the Preface to the fourth edition A few references to it may be found useful.

Prob 10. see p 22, Scholium A.

Prob 59 see p 32, Art 18.

Prob 101 and Prob 204 These are particular cases of a general theorem, for which see Art 120, Cor 2, or Art 144

Prob 150 This is proved by the method of infinitesimals see Art 32

Prob 395, note see Art 142 and Art 129

Prob 401 see under *Mechanical proofs of geometrical theorems*

Most of the foot-notes are taken from the same source

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